A Philosophical Foundation of Non-additive Measure and Probability

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INTERPRETING NON-ADDITIVITY OF SET FUNCTIONS

Idea: (cf. Murofushi and Sugeno [1989], [1990])

Non-additivity of a set function (fuzzy measure) μ expresses a relation between sets. Usually, relations between sets are modelled in a set theoretic way by intersection:

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Aim: Formalization of this idea

Method: Quasi analysis

THE QUASI ANALYSIS

Carnap [1923], [1928]

Analytic descriptions can be transformed into relation descriptions:

“X has the constituent/property x”

is formally equivalent to

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If the relation descrition is purely formal, then Carnap calls this process quasi analysis and the constituted classes quasi constituents or quasi properties.

The Quasi Analysis

Carnap [1923], [1928]

Quasi analysis is the process of constituting quasi constituents or quasi properties from a relation description holding over indivisible unities or propertyless points [cf. Carnap, 1928, §70].

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Carnap's motivation: Logical construction of the world

indivisible unities:	elementary experiences
relation description:	part similarity

THE QUASI ANALYSIS

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<u>The setting:</u>

Ω		set of basis elements
A	2^{Ω}	equivalence classes over Ω

Generally, the equivalence classes have to be derived from a given relation description.

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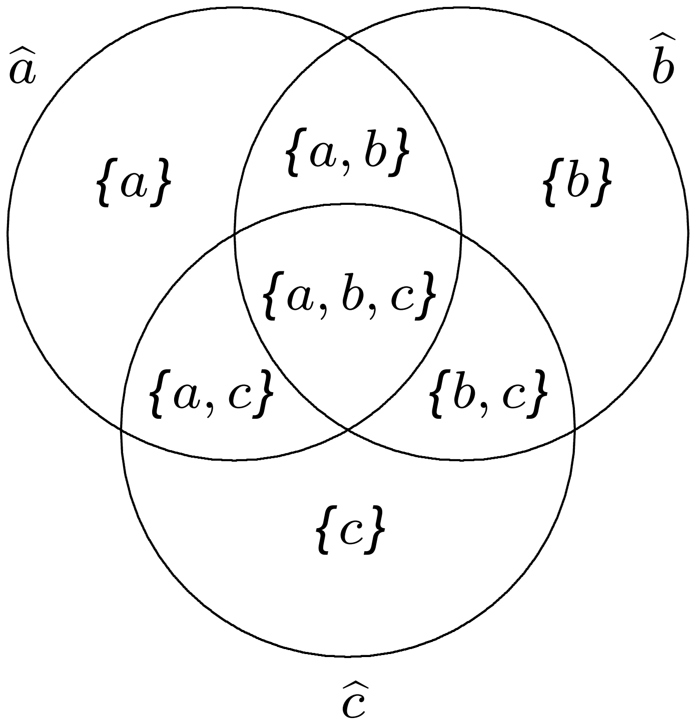
The quasi analysis:

A class A $\mathcal{A}$ is called a quasi property of each of its elements. If $\mathcal{A}$ contains enough sets, every ω can be identified with the collection of its quasi properties, $\widehat{\omega} := \{A \quad \mathcal{A} \mid \omega \quad A\}$.

For $\Omega := \{a, b, c\}$ and $A := 2^\Omega$ the quasi analysis can geometrically easily be illustrated:



Transforming Sets

Let Ω be a set and $\mathcal{A}$ a (σ-)algebra. Assign to every set of elements $A \quad \mathcal{A}$ the set of quasi properties of its elements:

$$\hat{A} := \{B \quad \mathcal{A} \mid \quad \omega \quad A : B \quad \omega\} = \{B \quad \mathcal{A} \mid B \cap A = \quad \}$$

Then for $A, B \quad \mathcal{A}$

(a) $\widehat{\ } = $

(b) $\widehat{\Omega} = \mathcal{A} \setminus \{\ \}$

(c) $A \quad B \Leftrightarrow \widehat{A} \quad \widehat{B}$

(d) $\widehat{A \quad B} = \widehat{A} \quad \widehat{B}$

(e) $\widehat{A \cap B} \quad \widehat{A} \cap \widehat{B}$

(f) $\widehat{A \setminus B} \quad \widehat{A} \setminus \widehat{B}$

Transforming Sets

Let Ω be a set and *A* a (σ-)algebra. Assign to every set of elements A *A* the set of quasi properties of its elements:

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(f) $\widehat{A \setminus B} \quad \widehat{A} \setminus \widehat{B}$

Remark:

(i) $A \cap B =$ implies $\widehat{A} \cap \widehat{B} =$, i.e. if A and B are related in a set theoretic way then $\widehat{A}$ and $\widehat{B}$ also are.

(ii) $A \quad B \quad \widehat{A} \cap \widehat{B}$ for all $A, B =$, i.e. even (non-empty) disjoint sets have joint quasi properties.

Transforming Set Functions

Theorem:

Let Ω be a finite set, and A an algebra over Ω.

(a) For any set function μ on A there exists a unique (signed) measure $\widehat{\mu}$ on the σ-algebra $\widehat{A} := \sigma\{\widehat{A} \mid A \quad A\}$ over $\widehat{\Omega}$ satisfying $\widehat{\mu}\left(\widehat{A}\right) = \mu(A)$ for all $A \quad A$.

(b) The measure $\widehat{\mu}$ is non-negative if and only if μ is totally alternating.

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(b) The measure $\widehat{\mu}$ is non-negative if and only if μ is totally alternating.

<u>Remark:</u> Part (a) implies

(i) $\mu(A) = \sum_{B \cap A =} \widehat{\mu}(\{B\})$,
i.e. μ measures the quasi properties of a set A and assigns this value to A.

(ii) $\widehat{\mu}(\widehat{A} \cap \widehat{B}) = \widehat{\mu}(\widehat{A}) + \widehat{\mu}(\widehat{B}) - \widehat{\mu}(\widehat{A} \quad \widehat{B}) = \mu(A) + \mu(B) - \mu(A \quad B)$,
i.e. our motivating idea of interpreting $\mu(A) + \mu(B) - \mu(A \quad B)$ as a measure of the strength of a relation between A and B is now formalized.

The following equations hold for any definition of $\widehat{f}$ and finite μ,

$$\begin{aligned}
\int_\Omega f\,d\mu &= \int_0^\infty \mu(\{f \geq x\})\,dx + \int_{-\infty}^0 \mu(\{f \geq x\}) - \mu(\Omega)\,dx \\
&= \int_0^\infty \widehat{\mu}(\widehat{\{f \geq x}\})\,dx + \int_{-\infty}^0 \widehat{\mu}(\{f \geq x\}) - \widehat{\mu}(\widehat{\Omega})\,dx, \\
\int_{\widehat{\Omega}} \widehat{f}\,d\widehat{\mu} &= \int_0^\infty \widehat{\mu}(\{\widehat{f} \geq x\})\,dx + \int_{-\infty}^0 \widehat{\mu}(\{\widehat{f} \geq x\}) - \widehat{\mu}(\widehat{\Omega})\,dx.
\end{aligned}$$

To obtain $\int_\Omega f\,d\mu = \int_{\widehat{\Omega}} \widehat{f}\,d\widehat{\mu}$, we need $\widehat{\{f \geq x\}} = \{\widehat{f} \geq x\}$. For A A,

$$A \quad \widehat{\{f \geq x\}} \qquad A \cap \{f \geq x\} = \qquad \overline{u_A}(f \geq x) = 1 \qquad \int_\Omega f d\overline{u_A} \geq x.$$

We therefore define $\widehat{f}(A) := \int_\Omega f d\overline{u_A}$ and obtain $\widehat{\{f \geq x\}} = \{\widehat{f} \geq x\}$. Hence,

$$\int_\Omega f\,d\mu = \int_{\widehat{\Omega}} \widehat{f}\,d\widehat{\mu}.$$

EXAMPLE: VOTING SYSTEMS

<u>The setting:</u>

$P_1, \ldots, P_n$	set of players
$2^{\{P_1,\ldots,P_n\}}$	set of possible coalitions
μ	characteristic function of the game,
	$\mu(C) := 1$ if C is a winning coalition, 0 else

<u>Definition:</u> Banzhaf Power Index for players, BPI(P):

$$\mathrm{BPI}(P) \;:=\; \frac{\#\text{ of times player P is critical}}{\#\text{ of times any player is critical}}$$

$$= \frac{\sum_{C\cap\{P\}=} \mu(C \quad \{P\}) - \mu(C)}{\sum_{i=1}^{n}\sum_{C\cap\{P\}=} \mu(C \quad \{P_i\}) - \mu(C)}$$

$$= \frac{\sum_{C\cap\{P\}=} \mu(\widehat{\{P\}} \setminus \widehat{C})}{\sum_{i=1}^{n}\sum_{C\cap\{P\}=} \mu(\widehat{\{P_i\}} \setminus \widehat{C})}$$

<u>Interpretation:</u> For any coalition C and any player P / C the quasi property $\widehat{\{P\}} \setminus \widehat{C}$ is the the marginal power contribution of P to C.

Example: Multi-Criteria Decision Making

cf. Grabisch [1996]

The setting (simplified):

$\{C_1, \ldots, C_n\}$ set of criterias

$a_1, \ldots, a_m : \{C_1, \ldots, C_n\} \to O$
set of acts mapping each criteria to an outcome $o \quad O$

Example:

$\{M, P, L\}$ set of subjects (mathematics, physics, literature)

$S_1, \ldots, S_n : \{M, P, L\} \to \{1, \ldots, 6\}$
set of student's grade functions

Aim: Rank students on the basis of their grades.

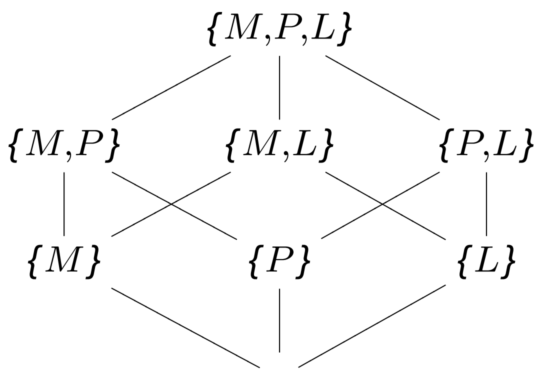
Standard method: Weight subjects and calculate the weighted mean.

Problem: Criterias have to be redundancy-free,
otherwise redundancies are overestimated.

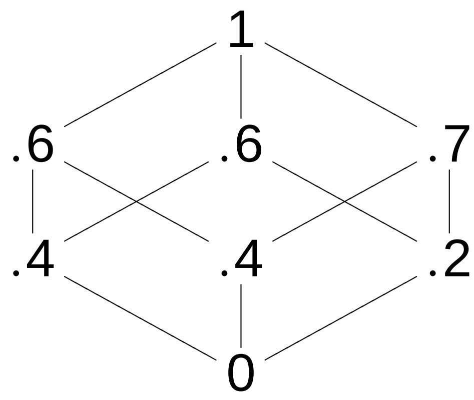
EXAMPLE: MULTI-CRITERIA DECISION MAKING

cf. Grabisch [1996]

Algebra:



Set function:



Function:

ω	M	P	L
$f(\omega)$	1	2	4

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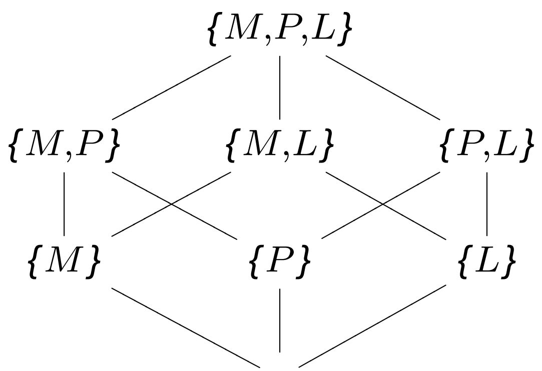

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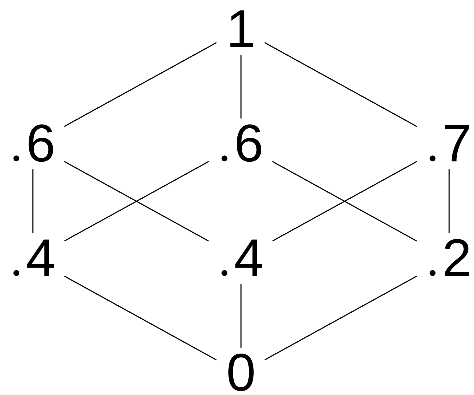

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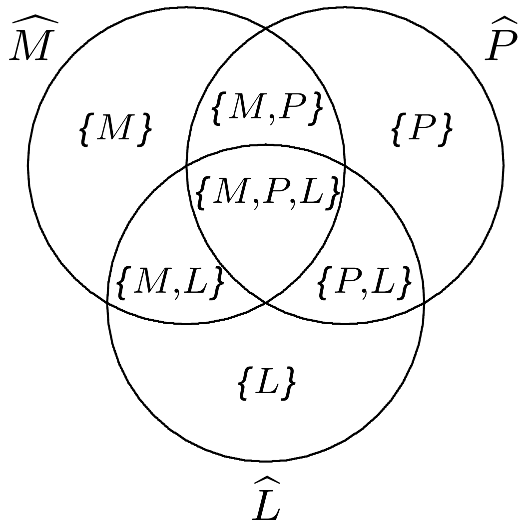
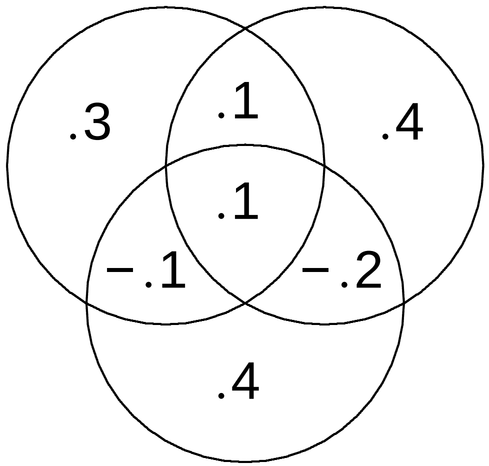
Transformed domain:



Transformed set function:



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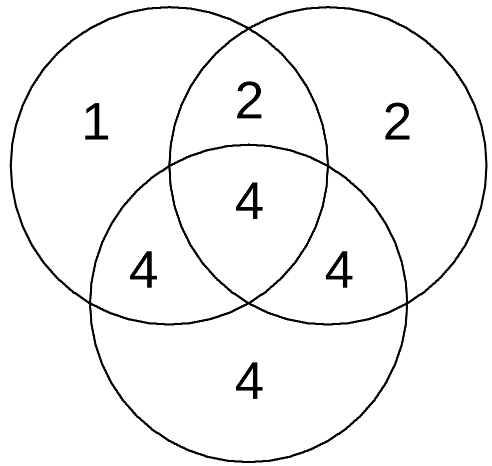

INDEPENDENCE

Definition: Two events A and B are called *(μ-)independent* iff $\widehat{A}$ and $\widehat{B}$ are ($\widehat{\mu}$-)independent,

$$\widehat{\mu}(\widehat{A} \cap \widehat{B}) = \widehat{\mu}(\widehat{A}) \cdot \widehat{\mu}(\widehat{B})$$

or, equivalently since $\widehat{\mu}(\widehat{A} \cap \widehat{B}) = \widehat{\mu}(\widehat{A}) + \widehat{\mu}(\widehat{B}) - \widehat{\mu}(\widehat{A \cup B})$, iff

$$\mu(A) + \mu(B) - \mu(A \cup B) = \mu(A) \cdot \mu(B).$$

Two random variables X and Y are called *(μ-)independent* iff $\widehat{X}$ and $\widehat{Y}$ are ($\widehat{\mu}$-)independent,

$$\widehat{\mu}\Big(\{\widehat{X \in U}\} \cap \{\widehat{Y \in V}\}\Big) = \widehat{\mu}\Big(\{\widehat{X \in U}\}\Big) \cdot \widehat{\mu}\Big(\{\widehat{Y \in V}\}\Big)$$

or, equivalently, iff

$$\mu(X \in U) + \mu(Y \in V) - \mu(X \in U \text{ or } Y \in V) = \mu(X \in U) \cdot \mu(Y \in V).$$

Conditioning and Products

<u>Definition:</u> Given a non-additive probability μ and two events A and B with $\mu(B) = 0$ the <u>conditional non-additive probability</u> of A given B is defined as the conditional probability of $\widehat{A}$ given $\widehat{B}$

$$\begin{aligned} \mu(A \mid B) &:= \widehat{\mu}(\widehat{A} \mid \widehat{B}) \\ &= \frac{\widehat{\mu}(\widehat{A} \cap \widehat{B})}{\widehat{\mu}(\widehat{B})} \\ &= \frac{\mu(A) + \mu(B) - \mu(A \quad B)}{\mu(B)}. \end{aligned}$$

Let μ_1 and μ_2 be two non-additive set functions.
Then their <u>product</u> $\mu_1 \quad \mu_2 : A_1 \times A_2 \to \mathbb{R}$ is defined by

$$\mu_1 \quad \mu_2(A \times B) := \widehat{\mu}_1 \quad \widehat{\mu}_2(\widehat{A \times B}).$$

Conclusion and Outlook

Non-additivity of set functions can often be explained by the interpretation that not all relations between sets have been modelled in a set theoretic way.

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The presented transform separates relations and evaluations and thus makes it possible to introduce concepts from standard measure and probability theory to the non-additive theory in a natural way.

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The presented transform separates relations and evaluations and thus makes it possible to introduce concepts from standard measure and probability theory to the non-additive theory in a natural way.

Result: Non-additivity does not necessarily mean a generalization of a theory originally built up on (σ-)additive measures.
It can just mean that the domain Ω of was chosen too small to express all possible relations that have to be modelled.