Exact functionals, functionals preserving linear inequalities, Lévy's metric

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Dissertation

zur Erlangung des Grades eines Doktors der Naturwissenschaften
– Dr. rer. nat. –

Vorgelegt im Fachbereich 3 (Mathematik & Informatik)
der Universität Bremen
im Juli 2003
Revidierte Fassung September 2003

Datum des Promotionskolloquiums: 8.10.2003

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Prof. Dr. Marc Keßeböhmer (Universität Bremen)

To Sonja and my parents

Preface

This thesis consists of three loosely related parts. In the first part, we examine a class of non-linear functionals which we call exact. The second part is devoted to a continuous linear representation of non-additive set functions and non-linear functionals. Finally, in the third part, we provide an estimate of the speed of convergence to the normal distribution in the Central Limit Theorem which improves existing results.

functions, such a result is already known as “basic probability assignment” in the Dempster-Shafer theory of evidence in the discrete case or as Möbius transform in the general case. We provide a non-linear homeomorphism between an arbitrary linear space of non-linear functionals with a distinguished convex subset on the one hand and a linear space of continuous linear functionals on the other hand such that the convex set is mapped onto the set of normalized monotone linear functionals. We pick up an idea first stated by Shafer in 1979 using Choquet’s Theorem for introducing his transform. Since the presented result does not depend on special properties of the domain of the functionals being transformed, also set functions are included in our investigations as they can be interpreted as functionals on the set of indicator functions. For the application of a special version of Choquet’s Theorem in our situation, it suffices to use a very general property shared by a number of classes of functionals like most classes of cooperative games or the class of coherent risk measures – they can be characterized as functionals preserving certain linear inequalities. Our result allows to switch between the non-additive theories and the much more elaborated theories of linear functionals like integration theory and functional analysis. Some outlines of these results have already been presented at the international conferences RUD 2003 and ISIPTA ’03.

In the third chapter, we deal with a classical problem in probability theory by estimating the speed of convergence to the normal distribution in the Central Limit Theorem. In contrast to the Berry–Esséen Theorem, we seem to be the first to use Lévy’s metric instead of Kolmogorov’s metric for this estimate while parallely avoiding characteristic functions. Unlike Kolmogorov’s metric, Lévy’s metric has not become very popular. Though it has a simple geometrical interpretation and can – in contrast to Kolmogorov’s metric – be used to metricize convergence in distribution in general, it has been almost completely disregarded in literature and got the status of being a curiosity, at best good enough for exercises in probability books. We provide some new estimates for Lévy’s metric which only use the absolute moments of the random variables involved. One of these estimates is used to obtain a Berry–Esséen type theorem and we show that our result in some cases improves the existing ones.

The work on this thesis was mainly supported by a grant of the Studienstiftung des deutschen Volkes. This support is gratefully acknowledged here. Finally, I would like to express my deep thanks to Dieter Denneberg for valuable discussions, helpful suggestions and comments.

Bremen, July 2003 Sebastian Maaß

Contents

Chapter 1

Exact Functionals and their Core

1.1 Introduction

Many classes of functionals and set functions are used to represent the different types of risk and uncertainty in utility, risk and decision theory. They show considerable similarities in their structure, their methods of use and the results obtained. The present chapter provides a general mathematical setting suitable for most approaches. We consider two classes of functionals which are defined on an *arbitrary* non-empty subset of the linear space $B(2^\Omega)$ of bounded real-valued functions. The class of *exact functionals* contains the classes of

- exact cooperative games (Schmeidler 1972 [35])
- coherent lower previsions (Walley 1991 [39])
- coherent risk measures (Artzner et al. 1999 [2]), (Delbaen 2002 [10])
- maxmin expected utility functionals (Gilboa and Schmeidler 1989 [19]).

The class of *exactifiable functionals* generalizes the classes of

- balanced cooperative games (Schmeidler 1972 [35])
- previsions avoiding sure loss (Walley 1991 [39]).

Methods and results of these different theories appear in a generalized form, relations between these theories are elaborated and several new results are presented. It turns out that the structural assumptions on the domain made in some theories mentioned above (e.g. being an algebra or a linear space) are mathematical unnecessary restrictions.

This chapter is organized as follows. In Section 2, exact and exactifiable functionals are defined and both types are characterized by a norm. For exact functionals, we present an extension theorem of Hahn-Banach type. Relations to the theories mentioned above are established in Section 3. Exact functionals will also be proved to be interpretable as a generalization of superadditive Choquet integrals where comonotonic additivity is relaxed to constant additivity. In Section 4, we introduce exact operators as a canonical method to transform the domains of exact functionals. In the special case where the range of an exact operator consists of bounded functions on a set of exact functionals, we obtain a result (Theorem 1.4.3) which can be interpreted as a general construction method for exact functionals. In the central Section 5, exact and exactifiable functionals are analyzed with functional analytical methods. The core concept mainly known from cooperative game theory is introduced for functionals and serves simultaneously as a basis for an analysis with methods from measure and integration theory. In the special case of exact Choquet integrals, there arises an interesting connection between the core of the integral and the core of the corresponding set function. Finally, we show that analogously to game theory continuity properties of exact functionals correspond directly to those of the elements of the core.

1.2 Definition and basic properties

Throughout this chapter, Ω denotes a non-empty set, 2^Ω the power set of Ω, $\mathcal{A}$ an algebra in 2^Ω, $B(\mathcal{A})$ the Banach space spanned by the indicator functions $\{1_A | A \in \mathcal{A}\}$ with the sup norm $\|\cdot\|_\infty$ and M a non-empty subset of $B(2^\Omega)$. A real-valued functional Γ on a linear space $S \subset B(2^\Omega)$ is called *superlinear* if it is *superadditive* (i.e. $\Gamma(f+g) \geq \Gamma(f) + \Gamma(g)$) and positively homogeneous. It is called *constant additive* if $\Gamma(f+c) = \Gamma(f) + \Gamma(c)$ for all $c, f \in S$, c constant. Constant additivity is widely denoted as translation invariance but we prefer the first notation since this property is similar to comonotonic additivity of the Choquet integral.

Definition 1.2.1 *Let* Γ *be the restriction to* M *of a monotone, superlinear, constant additive functional* $\Gamma' : B(2^\Omega) \to \mathbb{R}$*. Then* Γ *is called* ***exact*** *and* Γ' *an exact extension of* Γ*.*
Let $\Gamma : M \to \mathbb{R}$ *be a functional that can be dominated by an exact functional* $\Gamma' : M \to \mathbb{R}$*. Then* Γ *is called* ***exactifiable*** *and* Γ' *an exactification of* Γ*.*

For a real-valued functional $\Gamma : B(\mathcal{A}) \to \mathbb{R}$, one can define the conjugate functional $\overline{\Gamma} : B(\mathcal{A}) \to \mathbb{R}$ by $\overline{\Gamma}(f) := -\Gamma(-f)$. This definition is analogous to that of conjugate set functions. If Γ is a Choquet integral then $\overline{\Gamma}$ is the Choquet integral w.r.t. the conjugate set function. By this conjugation we get the dual theory of monotone, sublinear, constant additive functionals.

The condition of constant additivity in the definition of exactness can be expressed in different equivalent forms.

Proposition 1.2.2 *Let* $\Gamma : B(\mathcal{A}) \to \mathbb{R}$ *be a superlinear functional. Equivalent are*

(a) Γ *is constant additive,*

(b) $\Gamma(c) = c\Gamma(1)$ *for all constants* $c \in B(\mathcal{A})$*,*

(c) $\Gamma(1) = -\Gamma(-1)$*.*

Proof. Suppose (a), i.e. Γ is constant additive. For non-negative c, (b) holds by positive homogeneity of Γ. For $c < 0$ we have

$$0 = \Gamma(c - c) = \Gamma(c) + \Gamma(-c) = \Gamma(c) + -c\Gamma(1)$$

and therefore $\Gamma(c) = c\Gamma(1)$ for all constants $c \in B(\mathcal{A})$.
Now suppose (b). Then $\Gamma(1) = -\Gamma(-1)$ follows directly by setting $c := -1$.
Finally, suppose (c). Superlinearity of Γ yields $\Gamma(f+c) \geq \Gamma(f) + |c|\,\Gamma(\text{sign } c)$ and $\Gamma(f) \geq \Gamma(f + c) + |c|\,\Gamma(-\text{sign } c)$, hence equivalently

$$\Gamma(f) - |c|\,\Gamma(-\text{sign } c) \geq \Gamma(f + c) \geq \Gamma(f) + |c|\,\Gamma(\text{sign } c).$$

Thus equality, i.e. constant additivity, follows from $\Gamma(1) = -\Gamma(-1)$. □

It is easy to prove that monotone linear functionals as well as the infimum inf on $B(2^\Omega)$ are exact. Due to the one-to-one correspondence between sets and their indicator functions we identify set functions with functionals on indicator functions. By this means, we will call a set function exact if the corresponding functional on the set of indicator functions is exact.

We now define two norm-type functions (norms for short) closely related to the class of exact functionals. The first one is a generalization of a norm introduced for cooperative games by Schmeidler in (Schmeidler 1972 [35]) and will be proved in Theorem 1.2.6 to characterize the class of exactifiable functionals. The second norm characterizes the class of exact functionals (cf. Theorem 1.2.5). Actually, both "norms" are only Minkowski-functionals[1] but since in cooperative game theory the corresponding Minkowski-functionals are also called "norms" we follow the customary notation. For an arbitrary real-valued functional $\Gamma : M \to \mathbb{R}$, we therefore define

$$|\Gamma| \;:=\; \sup\left\{ \sum_{i=1}^{n} \lambda_i \Gamma(f_i) \;\middle|\; \sum_{i=1}^{n} \lambda_i f_i \le 1, n \in \mathbb{N}, \lambda_i \ge 0, f_i \in M \right\} \quad (1.1)$$

$$\|\Gamma\| \;:=\; \inf\left\{ c \in \mathbb{R}_+ \;\middle|\; \forall\, n \in \mathbb{N}, \lambda_i \ge 0, \lambda_0 \in \mathbb{R}, f, f_i \in M : \; f \ge \sum_{i=1}^{n} \lambda_i f_i + \lambda_0 \;\Rightarrow\; \Gamma(f) \ge \sum_{i=1}^{n} \lambda_i \Gamma(f_i) + \lambda_0 c \right\} \quad (1.2)$$

We now provide some rather elementary properties of the norms.

Proposition 1.2.3 *Let Γ be a functional on a non-empty set $M \subset B(2^\Omega)$.*

(a) $|\Gamma|, \|\Gamma\| \in [0, \infty]$.

(b) $|\lambda\Gamma| = \lambda|\Gamma|$ *and* $\|\lambda\Gamma\| = \lambda\|\Gamma\|$ *for all* $\lambda \in \mathbb{R}_+$.

(c) $|\Gamma_1 + \Gamma_2| \le |\Gamma_1| + |\Gamma_2|$ *and* $\|\Gamma_1 + \Gamma_2\| \le \|\Gamma_1\| + \|\Gamma_2\|$.

(d) $\|\Gamma\| = 0 \Rightarrow \Gamma = 0$.

(e) $|\cdot|$ *is monotone, i.e.* $\Gamma_1 \le \Gamma_2$ *implies* $|\Gamma_1| \le |\Gamma_2|$.

[1] A functional p on a linear space is called Minowski-functional, if $p(0) = 0$, $p(\lambda x) = \lambda p(x)$ for all $\lambda \ge 0$, and $p(x + y) \le p(x) + p(y)$

(f) *If* $\|\Gamma\| < \infty$ *then the infimum in Equation (1.2) is attained and*

$$\Gamma(f) = \sup\left\{ \sum_{i=1}^n \lambda_i\Gamma(f_i) + \lambda_0\|\Gamma\| \;\middle|\; \sum_{i=1}^n \lambda_i f_i + \lambda_0 \le f, \right.$$
$$\left. n \in \mathbb{N}, \lambda_0 \in \mathbb{R}, \lambda_i \ge 0 \text{ and } f_i \in M \right\}. \quad (1.3)$$

(g) $|\Gamma| \le \|\Gamma\|$.

(h) $|\Gamma| \le |\Gamma'|$ *and* $\|\Gamma\| \le \|\Gamma'\|$ *if* Γ' *is an extension of* Γ.

Proof.

(a) – (e) These proofs consist of elementary calculations.

(f) Assume the infimum is not attained, i.e. there are $n \in \mathbb{N}$, $\lambda_i \ge 0$, $\lambda_0 \in \mathbb{R}$ and $f, f_i \in M$ with $f \ge \sum_{i=1}^n \lambda_i f_i + \lambda_0$ and $\Gamma(f) < \sum_{i=1}^n \lambda_i\Gamma(f_i) + \lambda_0\|\Gamma\|$. Then there exists a neighborhood U of $\|\Gamma\|$ such that for all $c \in U$, c is not in the set defined in (1.2). Therefore $\|\Gamma\|$ cannot be the infimum which proves our assumption to be wrong. Equation (1.3) then follows directly from Equation (1.2) using the fact that the inf is a min.

(g) For all $n \in \mathbb{N}$, $\lambda_i \ge 0$ and $f, f_i \in M$ with $\sum_{i=1}^n \lambda_i f_i \le 1$ we get

$$\begin{aligned} f &\ge \sum_{i=1}^n \lambda_i f_i + f - 1, \\ \Gamma(f) &\ge \sum_{i=1}^n \lambda_i\Gamma(f_i) + \Gamma(f) - \|\Gamma\|, \\ \|\Gamma\| &\ge \sum_{i=1}^n \lambda_i\Gamma(f_i). \end{aligned}$$

(h) The assertion follows directly from the definitions of the norms. □

For an exact functional, both norms coincide with the operator norm if $1 \in M$. This condition holds in game theory (Ω is identified with its indicator function) or if $M = B(\mathcal{A})$.

Proposition 1.2.4 *For an exact functional $\Gamma : M \to \mathbb{R}$ with $1 \in M$, we have*

$$\|\Gamma\| = |\Gamma| = \Gamma(1) = \sup_{\substack{f \in M \\ \|f\|_\infty \neq 0}} \frac{|\Gamma(f)|}{\|f\|_\infty}. \tag{1.4}$$

Proof. By Proposition 1.2.3 (g) and the definition of the $|\cdot|$-norm, $\|\Gamma\| \geq |\Gamma| \geq \Gamma(1)$. Let $\Gamma' : B(2^\Omega) \to \mathbb{R}$ be an exact extension of Γ. Then for all $\lambda_i \geq 0, \lambda_0 \in \mathbb{R}, f, f_i \in M$ with $f \geq \sum_{i=1}^n \lambda_i f_i + \lambda_0$

$$\Gamma(f) = \Gamma'(f) \geq \Gamma'(\sum_{i=1}^n \lambda_i f_i + \lambda_0) \geq \sum_{i=1}^n \lambda_i \Gamma(f_i) + \lambda_0 \Gamma(1),$$

i.e. $\|\Gamma\| \leq \Gamma(1) < \infty$, hence we obtain $\|\Gamma\| = |\Gamma| = \Gamma(1)$. Obviously, $\Gamma(1) \leq \sup_{\|f\|_\infty \neq 0} \frac{|\Gamma(f)|}{\|f\|_\infty}$. For every $f \in M$ with $\|f\|_\infty \neq 0$, by exactness of Γ

$$\frac{|\Gamma(f)|}{\|f\|_\infty} = \left|\Gamma'\left(\frac{f}{\|f\|_\infty}\right)\right| \leq \Gamma(1).$$

This proves the last equation in (1.4). □

The following two theorems characterize, by means of our two norms, the class of exact, resp. exactifiable, functionals. Additionally, we obtain that the characterization of exact functionals in Equation (1.3) can be used to extend an exact functional in an analogous way as defining the inner set function or the inner measure known in (non-additive) measure theory[2].

Theorem 1.2.5 *Let Γ be a real-valued functional on a non-empty set M. Equivalent are*

(a) Γ *is exact.*

(b) $\|\Gamma\| < \infty$.

[2]It should be stressed that for an exact set function μ neither the exact extension μ_* defined in (1.5) nor the exact functional $\mu_\bullet$ defined in (1.8) (each restricted to 2^Ω) coincides with the inner set function known in non-additive measure theory or with the dual of the outer measure (i.e. $A \mapsto \mu(\Omega) - \mu^*$558.9664Tf15.991h-3850Td[(A)]T315.96Tf16.5101-3.809Td[c3

(c) The functional $\Gamma_* : B(2^\Omega) \to \mathbb{R}$ *defined by*

$$\Gamma_*(f) \quad := \quad \sup\left\{ \sum_{i=1}^n \lambda_i \Gamma(f_i) + \lambda_0 \|\Gamma\| \;\middle|\; \sum_{i=1}^n \lambda_i f_i + \lambda_0 \leq f, \right.$$
$$\left. n \in \mathbb{N}, \lambda_0 \in \mathbb{R}, \lambda_i \geq 0 \text{ and } f_i \in M \right\} \tag{1.5}$$

is an exact extension of Γ *with* $\|\Gamma_*\| = \|\Gamma\|$.

The equivalence relation (a) $\Leftrightarrow$ (b) in Theorem 1.2.5 shows that the definition of exactness does not rely on structural assumptions on the domain (which seems to be the case when defining exactness via functionals on a linear space) but only on the relations between the values of the functional (when calculating the $\|\cdot\|$-norm).

To simplify notations we omit some of the restrictions for the sup in the sequel if they are the same as in Equation (1.5)).

Proof of Theorem 1.2.5. We prove $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (a)$. Suppose Γ is exact. Then there exists an exact extension $\Gamma' : B(2^\Omega) \to \mathbb{R}$ of Γ and by Proposition 1.2.3 (h) and Proposition 1.2.4, $\|\Gamma\| \leq \|\Gamma'\| = \Gamma'(1) < \infty$. Now suppose (b), i.e. $\|\Gamma\| < \infty$. Monotonicity and superlinearity of Γ_* are easily verified from elementary properties of sup. Γ_* is real-valued because $\Gamma_*(f) \geq \|\Gamma\| \inf f > -\infty$ by definition of Γ_* and, since $0 = \Gamma_*(f-f) \geq \Gamma_*(f) + \Gamma_*(-f)$ by superlinearity of Γ_*, $\Gamma_*(f) \leq -\Gamma_*(-f) \leq -\|\Gamma\| \inf -f < \infty$ for all $f \in B(2^\Omega)$. By setting $f = 1$, $\lambda_0 = 1$ (resp. $f = -1$, $\lambda_0 = -1$) and the rest to zero in the definition of Γ_*, we have

$$\Gamma_*(1) \geq \|\Gamma\|, \quad \Gamma_*(-1) \geq -\|\Gamma\|, \tag{1.6}$$

and therefore, using superlinearity of Γ_*,

$$0 = \Gamma_*(1-1) \geq \Gamma_*(1) + \Gamma_*(-1) \geq \|\Gamma\| - \|\Gamma\| = 0, \tag{1.7}$$

hence $\Gamma_*(1) = -\Gamma_*(-1)$, i.e. Γ_* is constant additive by Proposition 1.2.2. Therefore Γ_* is an exact functional with $\|\Gamma_*\| = \Gamma_*(1) = \|\Gamma\|$ by Proposition 1.2.4 and Inequalities (1.6) and (1.7). By Proposition 1.2.3 (f), Γ_* extends Γ.

Finally, suppose (c), i.e. Γ_* is an exact extension of Γ. Then Γ is exact itself by Definition 1.2.1. □

The subsequent theorem characterizes exactifiable functionals in an analogous way as exact functionals are characterized in Theorem 1.2.5.

Theorem 1.2.6 *Let Γ be a real-valued functional on M. Equivalent are*

(a) Γ *is exactifiable.*

(b) $|\Gamma| < \infty$.

(c) The functional $\Gamma_\bullet : B(2^\Omega) \to \mathbb{R}$ *defined by*

$$\Gamma_\bullet(f) := \sup \left\{ \sum_{i=1}^{n} \lambda_i \Gamma(f_i) + \lambda_0 |\Gamma| \;\middle|\; \sum_{i=1}^{n} \lambda_i f_i + \lambda_0 \leq f \right\}. \tag{1.8}$$

is an exactification of Γ *on* M *with* $|\Gamma_\bullet| = |\Gamma|$.

Proof. We prove $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (a)$. Suppose Γ is exactifiable. Then there exists an exact functional $\Gamma' : B(2^\Omega) \to \mathbb{R}$ with $\Gamma' \geq \Gamma$. By Propositions 1.2.3 (h) and 1.2.4, $\infty > \Gamma'(1) = |\Gamma'|$ and by monotonicity of $|\cdot|$ we get $|\Gamma'| \geq |\Gamma|$, thus $|\Gamma| < \infty$. Now suppose (b), i.e. $|\Gamma| < \infty$. The proof of $\Gamma_\bullet$ being exact and $|\Gamma_\bullet| = |\Gamma|$ runs analogously to that of Theorem 1.2.5 only replacing $\|\cdot\|$ by $|\cdot|$. Additionally, $\Gamma_\bullet$ dominates Γ by definition. Finally, suppose (c), i.e. $\Gamma_\bullet$ is an exact functional dominating Γ on M. Then Γ is exactifiable by Definition 1.2.1. □

We now show that $\|\cdot\|$ and exact functionals, resp. $|\cdot|$ and exactifiable functionals, are closely related.

Proposition 1.2.7 *Let Γ be a real-valued functional on M.*

(a) If Γ *is exact then*

$$\Gamma_* = \inf \left\{ \Gamma' \;\middle|\; \Gamma' \textit{ is an exact extension of } \Gamma \textit{ with } \|\Gamma'\| = \|\Gamma\| \right\} \tag{1.9}$$

$$\|\Gamma\| = \inf \left\{ \|\Gamma'\| \;\middle|\; \Gamma' \textit{ is an exact extension of } \Gamma \right\}. \tag{1.10}$$

(b) If Γ *is exactifiable then*

$$\Gamma_\bullet = \inf \left\{ \Gamma' \;\middle|\; \Gamma' \textit{ is an exactification of } \Gamma \textit{ with } |\Gamma'| = |\Gamma| \right\} \tag{1.11}$$

$$|\Gamma| = \inf \left\{ |\Gamma'| \;\middle|\; \Gamma' \textit{ is an exactification of } \Gamma \right\}. \tag{1.12}$$

Proof.

(a) By Theorem 1.2.5, we have that Γ_* is contained in the set of functionals introduced in (1.9). Let $\Gamma' : B(2^\Omega) \to \mathbb{R}$ be an exact extension of Γ with $\|\Gamma'\| = \|\Gamma\|$. Then for every $f \in B(2^\Omega)$ applying Proposition 1.2.3 (f)

$$\begin{aligned}\Gamma'(f) &= \sup\left\{ \sum_{i=1}^n \lambda_i \Gamma'(f_i) + \lambda_0\|\Gamma'\| \;\middle|\; \sum_{i=1}^n \lambda_i f_i + \lambda_0 \leq f, f_i \in B(2^\Omega)\right\}\\ &\geq \sup\left\{ \sum_{i=1}^n \lambda_i \Gamma(f_i) + \lambda_0\|\Gamma\| \;\middle|\; \sum_{i=1}^n \lambda_i f_i + \lambda_0 \leq f, f_i \in M\right\}\\ &= \Gamma_*(f).\end{aligned}$$

For every exact extension $\Gamma' : B(2^\Omega) \to \mathbb{R}$ of Γ, by Proposition 1.2.3 (h), $\|\Gamma\| \leq \|\Gamma'\|$. The infimum is attained since $\|\Gamma\| = \|\Gamma_*\|$ by Theorem 1.2.5.

(b) By Theorem 1.2.6, we have that $\Gamma_\bullet$ is contained in the set of functionals introduced in (1.11). Let $\Gamma' : B(2^\Omega) \to \mathbb{R}$ be an exact functional dominating Γ on M with $|\Gamma'| = |\Gamma|$. Then analogously to (a), for every $f \in B(2^\Omega)$ additionally using $\|\Gamma'\| = |\Gamma'|$ by Proposition 1.2.4,

$$\begin{aligned}\Gamma'(f) &= \sup\left\{ \sum_{i=1}^n \lambda_i \Gamma'(f_i) + \lambda_0|\Gamma'| \;\middle|\; \sum_{i=1}^n \lambda_i f_i + \lambda_0 \leq f, f_i \in B(2^\Omega)\right\}\\ &\geq \sup\left\{ \sum_{i=1}^n \lambda_i \Gamma(f_i) + \lambda_0|\Gamma| \;\middle|\; \sum_{i=1}^n \lambda_i f_i + \lambda_0 \leq f, f_i \in M\right\}\\ &= \Gamma_\bullet(f).\end{aligned}$$

By Proposition 1.2.3 (h), we have $|\Gamma| \leq |\Gamma'|$ for every exactification $\Gamma' : B(2^\Omega) \to \mathbb{R}$ of Γ. The infimum is attained since $|\Gamma| = |\Gamma_\bullet|$ by Theorem 1.2.6. □

The condition of admitting only functionals having the same norms in Equations (1.9) and (1.11) in Proposition 1.2.7 cannot be omitted because there does not exist a minimal exact functional dominating a given functional Γ in general. This is implied by the fact that the infimum of exact functionals with different $\|\cdot\|$-norm is not constant additive as will be shown in the next example. Additionally, we show that $\Gamma_\bullet$ and Γ_* do not coincide in general even for exact functionals[3].

Example 1.2.8 *Let $M = \{-1\}$ and $\Gamma : M \to \mathbb{R}$ be defined by $\Gamma(-1) := -1$. Then Γ is exact since $\|\Gamma\| = 1$ and we have $|\Gamma| = 0$. Thus $\Gamma_\bullet = 0$ and $\Gamma_* = \inf$. Suppose, there were a minimal exact functional $\Gamma' : B(2^\Omega) \to \mathbb{R}$ dominating Γ on M. Then $\Gamma' \leq \Gamma_\bullet$ and $\Gamma' \leq \Gamma_*$ which implies $\Gamma'(-1) \leq -1$ and $\Gamma'(1) \leq 0$. Hence, Γ' would not be constant additive and therefore not exact – a contradiction.*

Since exact functionals having the same $\|\cdot\|$-norm will play an important role for further examinations, we call these functionals *equinormed*.

The functionals Γ_*, resp. $\Gamma_\bullet$, are of great importance for the following analysis of a given exact, resp. exactifiable, functional Γ as they have a domain with more structure than Γ. This allows us to demonstrate some properties of Γ by investigating Γ_*, resp. $\Gamma_\bullet$, particularly using functional analytical methods. Hence these functionals will be denoted as follows.

Definition 1.2.9 *For an exact functional $\Gamma : M \to \mathbb{R}$, Γ_* is called the* ***natural extension*** *of Γ.*
For an exactifiable functional $\Gamma : M \to \mathbb{R}$, $\Gamma_\bullet$ is called the ***natural exactification*** *of Γ.*

Next, we prove two inequalities for Γ in terms of $\|\cdot\|$ (cf. Walley 1991 [39, 2.6.1] and Delbaen 2002 [10, p. 4] for $\|\Gamma\| = 1$).

Proposition 1.2.10 *Let Γ be an exact functional on M. Then for all $f, g \in M$*

(a) $\|\Gamma\| \inf f \leq \Gamma(f) \leq \|\Gamma\| \sup f$,

(b) $|\Gamma(f) - \Gamma(g)| \leq \|\Gamma\| \cdot \|f - g\|_\infty$. *(Lipschitz-continuity)*

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[3] A sufficient condition for coincidence of $\Gamma_\bullet$ and Γ_* is that M contains a positive constant function (cf. Proposition 1.2.4).

Proof.

(a) From $f \geq \inf f$ we get $\Gamma(f) \geq \|\Gamma\| \inf f$ by definition of $\|\cdot\|$. Reversely, we get $\Gamma(f) \leq \|\Gamma\| \sup f$ from

$$-\Gamma(f) = -\Gamma_*(f) \geq \Gamma_*(-f) \geq \|\Gamma_*\| \inf -f = -\|\Gamma\| \sup f.$$

(b) $f = g + f - g \leq g + \|f - g\|_\infty$ implies $\Gamma(f) = \Gamma_*(f) \leq \Gamma_*(g + \|f - g\|_{\infty}$ [illegible] function μ, let $I_\mu f$ denote the Choquet integral

$$\int f\,d\mu = \int_{-\infty}^{0} \mu(f \geq x) - \mu(\Omega)\,dx + \int_{0}^{\infty} \mu(f \geq x)\,dx \tag{1.13}$$

w.r.t. μ for all $f \in B(\mathcal{A})$.

Proposition 1.3.1 *An exact functional Γ on $B(\mathcal{A})$ is representable as a Choquet integral if and only if it is comonotonic additive. A Choquet integral w.r.t. a set function μ is exact if and only if μ is supermodular.*

Proof. Comonotonic additivity is necessary for the representation as a Choquet integral. Sufficiency is implied by the theorem from Schmeidler in (Schmeidler 1986 [36]). The second equivalence is proved in Proposition 3 in the same paper. □

An exact functional being not representable as a Choquet integral can be constructed in the following way: Let $\emptyset \neq A \subsetneqq B \subsetneqq \Omega$ and $\Gamma : \{1_A, 1_B, 1_A + 1_B\} \to \mathbb{R}$ fulfill $2 \geq \Gamma(1_A + 1_B) > \Gamma(1_A) + \Gamma(1_B) \geq 0$ and $1 \geq \Gamma(1_B) \geq \Gamma(1_A)$. Then Γ is exact but – like Γ_* – not comonotonic additive. Hence, Γ is not representable as a Choquet integral.

Supermodular set functions are a very remarkable class in the presented theory because in Corollary 1.3.2 and Proposition 1.3.3 we will obtain that not only they are exact itself but also the corresponding Choquet integral coincides with their natural extension (cf. Krätschmer 2003 [24, Theorem 5.2]).

Corollary 1.3.2 *Supermodular set functions $\mu : \mathcal{A} \to \mathbb{R}_+$ are exact.*

Proof. If μ is a supermodular set function then I_μ is an extension of μ which is exact by Proposition 1.3.1. □

Proposition 1.3.3 *Let μ be an exact set function on an algebra $\mathcal{A}$. Then*

$$\mu_* | B(\mathcal{A}) \geq I_\mu \tag{1.14}$$

and equality holds if and only if μ is supermodular.

Proof. It is sufficient to prove the assertion for simple functions due to the continuity properties of exact functionals (cf. Proposition 1.2.10 (b)) and Choquet integrals. For any simple function f on $B(\mathcal{A})$ in standard form, i.e. $f = \sum_{i=1}^n \lambda_i 1_{A_i} + \lambda_0$ with $\lambda_0 \in \mathbb{R}$, $\lambda_i \geq 0, A_i \in \mathcal{A}, A_i \subset A_j, i \leq j$ we have

$$\begin{aligned} I_\mu(f) &= \sum_{i=1}^n \lambda_i \mu(A_i) + \lambda_0 \mu(\Omega) \\ &\leq \sup \left\{ \sum_{i=1}^n \lambda_i' \mu(B_i) + \lambda_0' \|\mu\| \;\middle|\; \sum_{i=1}^n \lambda_i' 1_{B_i} + \lambda_0' \leq f \right\} \\ &= \mu_*|B(\mathcal{A})(f). \end{aligned}$$

For equality, it is necessary that I_μ is exact since $\mu_*|B(\mathcal{A})$ is exact. By Proposition 1.3.1 this is equivalent to supermodularity of μ. Reversely, if μ is supermodular we have $\mu(A) = \mu_*|B(\mathcal{A})(1_A)$ for all $A \in \mathcal{A}$ by Corollary 1.3.2 and $\mu(A) = I_\mu(1_A)$ for all $A \in \mathcal{A}$. Additionally, by Proposition 1.2.4, $\|\mu_*\| = \|\mu\| = \|I_\mu\|$ and using Proposition 1.2.7 (a) yields $\mu_*|B(\mathcal{A}) \leq I_\mu$. □

We now investigate the relation between exact, resp. exactifiable, functionals and cooperative game theory. A cooperative game v is a bounded, non-negative, real-valued set function on an algebra $\mathcal{A}$ over Ω, mapping the empty set to 0. Two classes of cooperative games are of special interest here, the *balanced games* and the *exact games.*

A cooperative game v is called balanced if for all $n \in \mathbb{N}, \lambda_i \geq 0, A_i \in \mathcal{A}$

$$\sum_{i=1}^n \lambda_i 1_{A_i} \leq 1_\Omega \quad \Longrightarrow \quad \sum_{i=1}^n \lambda_i v(A_i) \leq v(\Omega). \tag{1.15}$$

A cooperative game v is called exact if for all $n \in \mathbb{N}, \lambda_0, \lambda_i \geq 0, A, A_i \in \mathcal{A}$

$$\sum_{i=1}^n \lambda_i 1_{A_i} - \lambda_0 \leq 1_A \quad \Longrightarrow \quad \sum_{i=1}^n \lambda_i v(A_i) - \lambda_0 |v| \leq v(A). \tag{1.16}$$

Proposition 1.3.4 *Let v be a non-negative set function on an algebra $\mathcal{A}$.*

(a) v is a balanced cooperative game if and only if v is exactifiable with $v(\Omega) = |v|$.

(b) v is an exact cooperative game if and only if v is exact.

Proof.

(a) By definition of $|\cdot|$, a game v is balanced if and only if $v(\Omega) = |v|$ (cf. Schmeidler 1972 [35, Corollary 2.4]). The assertion then follows from Theorem 1.2.6 since $v(\Omega) < \infty$.

(b) Suppose v is an exact cooperative game. By setting $\lambda_0 = 0$ and $A = \Omega$ in (1.16) we obtain $v(\Omega) = |v| < \infty$. Therefore, using the definition of $|\cdot|$, implication (1.16) remains true when admitting negative λ_0, thus $|v| \geq \|v\|$ by definition of $\|\cdot\|$. The reverse inequality holds by Proposition 1.2.3 (g), hence $\|v\| = |v| < \infty$ and therefore v is exact by Theorem 1.2.5.
Now suppose v is exact in the sense of Definition 1.2.1. Applying Proposition 1.2.4, implication (1.16) holds by definition of $\|\cdot\|$, thus v is an exact cooperative game. □

To show that balanced cooperative games are a proper subclass of exactifiable set functions, let v be defined on $2^{\{1,2\}}$ by $v(A) = 1 \Leftrightarrow A \neq \emptyset$. Then $|v| = 2$, hence v is exactifiable by Theorem 1.2.6 but $v(\{1,2\}) < |v|$.

Proposition 1.3.4 shows that the definitions of balanced, resp. exact, cooperative games do not rely on structural assumptions on the domain, i.e. supposing the domain being an algebra (cf. first paragraph following Theorem 1.2.5). These mathematically unnecessary, restrictive assumptions are primarily motivated by applications and do not influence the results obtained. But assuming a rich structure of the domain and using Proposition 1.2.4 the definition of exactness can more easily be expressed using this structure as it is done usually.

A well-known result in cooperative game theory is that balanced, resp. exact, cooperative games can be characterized by properties of the set of additive set functions dominating the game and having the same $|\cdot|$-norm, i.e. the core of the game. These relations remain true in our more general context and will be analyzed in Section 5.

We now take a look at the relation of our classes of functionals to the theory of imprecise previsions. Walley examined in (Walley 1991 [39]) mainly two classes of functionals on an arbitrary non-empty subset of $B(2^\Omega)$ to model rational behaviour in decision situations. These are the *lower previsions avoiding sure loss* and the *coherent lower previsions.*

A real-valued functional Γ on a non-empty subset M of $B(2^\Omega)$ is called a lower prevision avoiding sure loss (cf. Walley 1991 [39, Definition 2.4.1 and Lemma 2.4.4]) if for all $n \in \mathbb{N}$, $\lambda_i \geq 0, f_i \in M$

$$\sup \sum_{i=1}^{n} \lambda_i f_i \geq \sum_{i=1}^{n} \lambda_i \Gamma(f_i). \tag{1.17}$$

A real-valued functional Γ on a non-empty subset M of $B(2^\Omega)$ is called a coherent lower prevision (cf. Walley 1991 [39, Definition 2.5.1 and Lemma 2.5.4]) if for all $n \in \mathbb{N}$, $\lambda_0, \lambda_i \geq 0,\ f_0, f_i \in M$

$$\sup \left(\sum_{i=1}^{n} \lambda_i f_i - \lambda_0 f_0 \right) \geq \sum_{i=1}^{n} \lambda_i \Gamma(f_i) - \lambda_0 \Gamma(f_0). \tag{1.18}$$

Proposition 1.3.5 *Let Γ be a real-valued functional on a non-empty set $M \subset B(2^\Omega)$.*

(a) Γ is a lower prevision avoiding sure loss if and only if it is exactifiable and there exists an exactification Γ' of Γ with $\|\Gamma'\| = 1$.

(b) Γ is a coherent lower prevision if and only if it is exact and there exists an exact extension of Γ with $\|\Gamma'\| = 1$.

Proof.

(a) Walley has proved in (Walley 1991 [39]) that Γ is a prevision avoiding sure loss if and only if

$$B(2^\Omega) \to \overline{\mathbb{R}},\ f \mapsto \sup \left\{ \sum_{i=1}^{n} \lambda_i \Gamma(f_i) + \lambda_0 \;\middle|\; \sum_{i=1}^{n} \lambda_i f_i + \lambda_0 \leq f \right\} \tag{1.19}$$

is real-valued (cf. Walley 1991 [39, p. 123]). This is equivalent to this functional being exact with $|\cdot|$-norm 1 by Theorem 1.2.6, hence the assertion holds.

(b) Γ is a coherent lower prevision if and only if the functional defined in (1.19) extends Γ (cf. Walley 1991 [39, Proposition 3.1.2 and Lemma 3.1.3]). Analogously to part (a), the assertion holds using Theorem 1.2.5. □

Walley has also defined a "natural extension" for coherent lower previsions (cf. Walley 1991 [39]). In contrast to the natural extension defined in this chapter, Walley's generally does not preserves the $\|\cdot\|$-norm of the coherent lower prevision.

Propositions 1.3.4 (b) and 1.3.5 (b) imply that on the one hand the intersection of the classes of exact games and coherent lower previsions consists of all exact games satisfying $v(\Omega) = 1$ and on the other hand the class of exact functionals is the smallest convex cone (cf. Proposition 1.2.3 (b), (c)) containing exact games and coherent lower previsions.

Finally, the relation of exact functionals to the theory of risk measures is outlined. Artzner et al. and Delbaen examined in (Artzner et al. 1999 [2]), resp. (Delbaen 2002 [10]) a class of risk measures which they call *coherent risk measures.*

A real-valued functional Γ on a linear space $B(\mathcal{A})$ is called a coherent risk measure if (cf. Delbaen 2002 [10, Definition 2.1])

(a) $\Gamma(f) \leq 0$ if $f \geq 0$,

(b) $\Gamma(f+g) \leq \Gamma(f) + \Gamma(g)$,

(c) $\Gamma(\lambda f) = \lambda\Gamma(f)$ for all $\lambda \geq 0$,

(d) $\Gamma(f+c) = \Gamma(f) - c$ for all $c \in \mathbb{R}$.

These functionals are the negatives of normalized exact functionals, i.e. $\Gamma : B(\mathcal{A}) \to \mathbb{R}$ is exact with $\|\Gamma\| = 1$ if and only if $-\Gamma$ is a coherent risk measure (cf. Maaß 2000 [28]). By Proposition 1.3.5 (b), the negatives of coherent risk measures are also coherent lower previsions.

1.4 Exact operators

In some situations, it might be useful to perform transformations on the domain of exact functionals. There are two natural ways of defining such transforms. One is to introduce it via a function $\varphi : \Omega_1 \to \Omega_2$ satisfying

$$f_2 \circ \varphi \in M_1 \quad \text{for all } f_2 \in M_2 \tag{1.20}$$

(cf. Maaß 2000 [28, p. 22]). If M_1, M_2 are σ-algebras then Condition (1.20) is equivalent to measurability of φ and if M_1, M_2 are topologies then Condition (1.20) is equivalent to continuity of φ. In the sequel, we will consider a second way of defining a transform, i.e. operators directly mapping M_1 to M_2. Therefore, we introduce exact operators in the following way.

Definition 1.4.1 *An operator $O : M_1 \to M_2$ is called **exact** if for each exact functional Γ on M_2 the functional $\Gamma \circ O$ is exact.*

Of course, we now have to characterize the exact operators in more concrete terms. This characterization is almost the same as that of exact functionals (cf. Theorem 1.2.5).

Proposition 1.4.2 *An operator $O : M_1 \to M_2$ is exact if and only if*

$$O(f) = \sup\left\{ \sum_{i=1}^{n} \lambda_i O(f_i) + \lambda_0 \|O\| \;\middle|\; \sum_{i=1}^{n} \lambda_i f_i + \lambda_0 \leq f, \; n \in \mathbb{N}, \lambda_0 \in \mathbb{R}, \lambda_i \geq 0 \text{ and } f_i \in M \right\} \quad (1.21)$$

with

$$\|O\| := \inf\left\{ c \in \mathbb{R}_+ \;\middle|\; \forall\, n \in \mathbb{N}, \lambda_i \geq 0, \lambda_0 \in \mathbb{R}, f, f_i \in M : f \geq \sum_{i=1}^{n} \lambda_i f_i + \lambda_0 \;\Rightarrow\; O(f) \geq \sum_{i=1}^{n} \lambda_i O(f_i) + \lambda_0 c \right\}. \quad (1.22)$$

Moreover, $\|\Gamma \circ O\| = \|\Gamma\| \cdot \|O\|$ holds.

Proof. If Equation (1.21) holds for O then, using Theorem 1.2.5, it is easy to show that $\Gamma \circ O$ is exact for every exact Γ and that $\|\Gamma \circ O\| = \|\Gamma\| \cdot \|O\|$ holds. Now suppose

$$O(f) \not\geq \sum_{i=1}^{n} \lambda_i O(f_i) + \lambda_0 \|O\| \quad (1.23)$$

with $\sum_{i=1}^n \lambda_i f_i + \lambda_0 \leq f$ for some $n \in \mathbb{N}$, $\lambda_0 \in \mathbb{R}$, $\lambda_i \geq 0$ and $f, f_i \in M$. Then, for some $\omega_2' \in \Omega_2$ with $O(f) < \sum_{i=1}^n \lambda_i O(f_i) + \lambda_0 \|O\|$ and for the exact (linear) functional $\Gamma_{\omega_2'} : M_2 \to \mathbb{R}$, $\Gamma_{\omega_2'}(f) := f(\omega_2')$ obviously

$$\Gamma_{\omega_2'}(O(f)) < \sum_{i=1}^n \lambda_i \Gamma_{\omega_2'}(O(f_i)) + \lambda_0 \|O\| \|\Gamma_{\omega_2'}\| \tag{1.24}$$

holds, i.e. $\Gamma \circ O$ is not exact. □

The following application of Proposition 1.4.2 provides a powerful construction method for exact functionals. Several results concerning the set of exact functionals can be obtained from this fairly general theorem. The main idea is to set the range space M_2 to a set of bounded functions on a set of exact functionals.

Theorem 1.4.3 *Let $\mathcal{G}$ be a non-empty set of exact functionals on $M \subset B(2^\Omega)$ being uniformly bounded, i.e. $\sup_{\Gamma' \in \mathcal{G}} \|\Gamma'\| < \infty$, and for every $f \in M$ let $\tilde{f} \in B(2^{\mathcal{G}})$ be defined by $\tilde{f}(\Gamma') := \Gamma'(f)$. Furthermore, let $\Gamma : B(2^{\mathcal{G}}) \to \mathbb{R}$ be an exact functional. If Γ is linear or if the elements in $\mathcal{G}$ are equinormed then the functional*

$$M \to \mathbb{R}, \quad f \mapsto \Gamma(\tilde{f}) \tag{1.25}$$

is exact.

Proof. The operator

$$O : M \to \mathbb{R}^{\mathcal{G}}, \quad O(f) := \tilde{f}$$

is well-defined because the function $\Gamma' \mapsto \Gamma'(f)$ is bounded for every $f \in M$ since $-\infty < \|\Gamma\| \inf f \leq \Gamma'(f) \leq \|\Gamma\| \sup f < \infty$ for all $\Gamma' \in \mathcal{G}$. First, assume that all Γ' in $\mathcal{G}$ are equinormed. Then

$$\begin{aligned} \sum_{i=1}^n \lambda_i f_i + \lambda_0 \leq f \quad &\Longrightarrow \quad \sum_{i=1}^n \lambda_i \Gamma'(f_i) + \lambda_0 \|\Gamma'\| \leq \Gamma'(f) \quad \forall\, \Gamma' \in \mathcal{G} \\ &\Longleftrightarrow \quad \sum_{i=1}^n \lambda_i O(f_i) + \lambda_0 \|\Gamma'\| \leq O(f) \end{aligned}$$

with $n \in \mathbb{N}$, $\lambda_0 \in \mathbb{R}$, $\lambda_i \geq 0$ and $f, f_i \in M$. By Proposition 1.4.2, O is exact with the same norm as all $\Gamma' \in \mathcal{G}$. Hence, the functional $\Gamma \circ O$ as defined in

Equation (1.25) is exact as a concatenation of an exact functional with the exact operator O.
Now allow that the exact functionals in $\mathcal{G}$ are not equinormed. Then O is not exact in general. Since the mapping

$$\mathcal{G} \to \{\Gamma'_* \mid \Gamma' \in \mathcal{G}\}, \quad \Gamma' \mapsto \Gamma'_*$$

is bijective, we can assume w.l.o.g. $M = B(2^\Omega)$, especially $\|\Gamma'\| = \Gamma'(1)$ (cf. Proposition 1.2.4). Then, by exactness of each $\Gamma' \in \mathcal{G}$,

$$\begin{aligned} O(f)(\Gamma') &= \Gamma'(f) \\ &\geq \sum_{i=1}^{n} \lambda_i \Gamma'(f_i) + \lambda_0 \Gamma'(1) \\ &= \sum_{i=1}^{n} \lambda_i O(f_i)(\Gamma') + \lambda_0 O(1)(\Gamma') \end{aligned}$$

if $f \geq \sum_{i=1}^{n} \lambda_i f_i + \lambda_0$ with $n \in \mathbb{N}$, $\lambda_0 \in \mathbb{R}$, $\lambda_i \geq 0$ and $f, f_i \in M$. For a linear exact functional $\Gamma : B(2^{\mathcal{G}}) \to \mathbb{R}$ we then obtain

$$\Gamma(O(f)) \geq \sum_{i=1}^{n} \lambda_i \Gamma(O(f_i)) + \lambda_0 \Gamma(O(1)),$$

i.e. $\|\Gamma \circ O\| \leq \Gamma(O(1)) < \infty$ (cf. Equation (1.2)). Thus $\Gamma \circ O$ is exact by Theorem 1.2.5. Since restrictions of exact functionals are again exact, this result also holds if $M \neq B(2^\Omega)$. □

We now state some implications of Theorem 1.4.3 which are well-known, e.g. in the theory of imprecise previsions (cf. Walley 1991 [39, 2.6.3 - 2.6.7]), but their mathematical relations in the sense of being special cases of a very general construction method have not been mentioned yet.

Corollary 1.4.4 *Let $\{\Gamma_i\}_{i \in I}$ be a non-empty indexed set of equinormed exact functionals on $M \subset B(2^\Omega)$.*

(a) The lower envelope $\inf_{i \in I} \Gamma_i$ of the Γ_i is exact.

(b) Finite positive linear combinations in $\{\Gamma_i\}_{i \in I}$ are exact.

If, especially, $I = \mathbb{N}$ *then*

(c) The limit inferior $\liminf_{i\to\infty} \Gamma_i$ *of the* Γ_i *is exact.*

(d) If Γ_i *is a pointwise convergent sequence then the limit* $\lim_{i\to\infty} \Gamma_i$ *is exact.*

(e) If Γ_i *is an increasing sequence then the limit* $\sup_{i\to\infty} \Gamma_i$ *is exact.*

Proof. The infimum, positive linear combinations, the limit inferior as well as the limit are exact functionals on $\{\Gamma_i\}_{i\in I}$. The assertions (a) – (d) then follow direct from Theorem 1.4.3. For (e) we additionally have to use that the limit exists since all $\Gamma_i(f)$ are bounded by $\|\Gamma_i\| \sup f$ for all $f \in M$. □

We conclude this section with a sketch of a possible application of Theorem 1.4.3. Suppose there are n experts assigning values "in a normalized exact way" to all gambles $f \in M$, i.e. $\mathcal{G} := \{\Gamma_i\}_{i=1,\dots,n}$ is a non-empty set of normalized exact functionals. Furthermore, suppose we also want to assign values in an exact way to all $f \in M$ just by incorporating the Γ_i. By Corollary 1.4.4 (a), we could take the lower envelope of all Γ_i, $\inf_{i=1,\dots,n} \Gamma_i$, as our exact functional if we were very cautious. If we had certain opinions on the exact functionals of all experts we also could assign weights $\lambda_i \geq 0$ to every Γ_i and take $\sum_{i=1}^{n} \lambda_i \Gamma_i$ as our exact functional (cf. Corollary 1.4.4 (b)). But using Theorem 1.4.3, we can go even further. For example, we can assign weights $\mu(A)$ to "coalitions" $A \subset \{1, \dots, n\}$ of experts in order to express that the unanimity of certain experts on the evaluation of some gamble f should count more than the unanimity of some other coalitions. If this set function μ is supermodular then the Choquet integral $\int \cdot \, d\mu$ is exact and, by Theorem 1.4.3, so is the lower prevision $f \mapsto \int (i \mapsto \Gamma_i(f)) \, d\mu$.

1.5 The core of functionals

In this section, we adopt the core concept from cooperative game theory to our theory of functionals on arbitrary subsets of $B(2^\Omega)$. Similar concepts are known in all theories mentioned in the introduction. The core allows us to analyze exact and exactifiable functionals with methods from functional analysis as well as measure and integration theory.

Throughout the remaining part of this chapter we identify the dual space $B^*(\mathcal{A})$ of $B(\mathcal{A})$ with the space of bounded additive set functions on $\mathcal{A}$, $ba(\mathcal{A})$, due to the existence of a natural isometric isomorphism between these spaces (cf. Dunford and Schwartz 1958 [14, Theorem IV.5.1]), i.e. linear functionals are sometimes interpreted as additive set functions and vice versa. An important subspace of $ba(\mathcal{A})$ used in this section is the space of bounded countably additive set functions on $\mathcal{A}$, $ca(\mathcal{A})$.

Definition 1.5.1 *Let $\Gamma : M \to \mathbb{R}$ be a functional and $\mathcal{A}$ an algebra satisfying $M \subset B(\mathcal{A})$. Then $\mathcal{A}$-core of Γ*

$$\mathcal{C}_{\mathcal{A}}(\Gamma) := \left\{ \Lambda \in B^*(\mathcal{A}) \mid \Lambda|M \geq \Gamma, \Lambda \text{ monotone}, |\Lambda| = |\Gamma| \right\} \tag{1.26}$$

*is called the **$\mathcal{A}$-core** of Γ. If no confusion about the algebra used is possible or a result concerning the $\mathcal{A}$-core does not depend on the algebra we call the $\mathcal{A}$-core just **core** and denote it by $\mathcal{C}(\Gamma)$.*

As noted in Section 1, the elements of the core are exact because they are monotone, linear and real-valued. We first state a simple result on the core.

Proposition 1.5.2 *Let $\Gamma : M \to \mathbb{R}$ be a functional and $\mathcal{A}$ an algebra satisfying $M \subset B(\mathcal{A})$. Then Γ is exactifiable if $\mathcal{C}_{\mathcal{A}}(\Gamma) \neq \emptyset$.*

Proof. If $\Lambda \in \mathcal{C}(\Gamma)$ then $|\Lambda| < \infty$ by Proposition 1.2.4, such that exactifiability of Γ is necessary for any $\Lambda \in B^*(\mathcal{A})$ satisfying the condition $|\Lambda| = |\Gamma|$ in Definition 1.5.1 and therefore the core being non-empty. □

The restriction to equinormed linear functionals in the definition of the core is necessary to apply Theorem 1.4.3, resp. Corollary 1.4.4, to the core for proving the main theorem in this section. It will turn out in Corollary 1.5.9 that this restriction does not influence the question under what conditions the core is non-empty.

Due to our identification of $B^*(\mathcal{A})$ with $ba(\mathcal{A})$ we have

$$\mathcal{C}_{\mathcal{A}}(\Gamma) = \left\{ \lambda \in ba(\mathcal{A}) \mid I_\lambda|M \geq \Gamma, \lambda \geq 0, \lambda(\Omega) = |\Gamma| \right\} \tag{1.27}$$

since $\lambda(\Omega) = |\lambda|$ and $|I_\lambda| = |\lambda|$ by exactness of λ, by Proposition 1.2.4 and Proposition 1.3.3.

For a cooperative game v on an algebra $\mathcal{A}$, the definition of the core given here corresponds to the one in cooperative game theory. The core of a cooperative game $v : \mathcal{A} \to \mathbb{R}_+$ is defined by

$$\operatorname{core}(v) := \big\{\lambda : \mathcal{A} \to \mathbb{R} \text{ additive} \mid \lambda \geq v, \lambda(\Omega) = v(\Omega)\big\}. \tag{1.28}$$

The definitions (1.27) and (1.28) of the core coincide if $v(\Omega) = |v|$, i.e. if v is balanced. If v is not balanced then $\lambda(\Omega) = |\lambda| \geq |v| > v(\Omega)$ for all monotone, additive λ dominating v, i.e. $\operatorname{core}(v) = \emptyset$. Hence $\operatorname{core}(v) \subset \mathcal{C}_{\mathcal{A}}(v)$ and, analogously to Proposition 1.5.2, $\operatorname{core}(v) \neq \emptyset$ implies v being balanced.

The following two propositions show the connection between set functions, their Choquet integrals and the corresponding cores. In Proposition 1.5.3, the cores of a set function and its Choquet integral are compared whereas in Proposition 1.5.4 (cf. Denneberg 1994 [11, Proposition 10.3]) we investigate the relation of the set function and the functional defined as the infimum of a non-empty equinormed subset of $B^*(\mathcal{A})$.

Proposition 1.5.3 *Let μ be a finite monotone set function on an algebra $\mathcal{A}$. Then*

$$\mathcal{C}(\mu) = \mathcal{C}(I_\mu). \tag{1.29}$$

Proof. For every $\lambda \in \mathcal{C}(\mu)$, we have $I_\lambda \in \mathcal{C}(I_\mu)$ because the definition of the Choquet integral implies $\lambda \geq \mu \Rightarrow I_\lambda \geq I_\mu$. Reversely, $\Lambda \in \mathcal{C}(I_\mu)$ implies $\Lambda|\mathcal{A} \in \mathcal{C}(\mu)$ trivially. □

Proposition 1.5.4 *Let $\mathcal{C}$ be a non-empty subset of $B^*(\mathcal{A})$ consisting of equinormed monotone functionals and let $\mu : \mathcal{A} \to \mathbb{R}_+$ be defined by $\mu(A) := \inf_{\Lambda \in \mathcal{C}} \Lambda(1_A)$. Then for all $f \in B(\mathcal{A})$*

$$\inf_{\Lambda \in \mathcal{C}} \Lambda(f) \geq I_\mu(f) \tag{1.30}$$

and equality holds if and only if μ is supermodular.

Proof. By using the isometric isomorphy of $B^*(\mathcal{A})$ and $ba(\mathcal{A})$, we obtain $\inf_{\Lambda \in \mathcal{C}} \Lambda = \inf_{\lambda \in \mathcal{C}} I_\lambda \geq I_\mu$. By Proposition 1.3.1, equality holds if and only if μ is supermodular. □

We now show that on the one hand the core of a functional and its exactification and on the other hand the different cores of a functional are essentially identical.

Proposition 1.5.5 *Let* Γ *be a real-valued functional on* M *and* $\mathcal{A}_1, \mathcal{A}_2$ *two algebras satisfying* $M \subset B(\mathcal{A}_1) \subset B(\mathcal{A}_2)$. *Then*

(a) $\mathcal{C}_{\mathcal{A}_1}(\Gamma) = \{\Lambda | B(\mathcal{A}_1) \mid \Lambda \in \mathcal{C}_{2^\Omega}(\Gamma_\bullet)\}$,

(b) $\mathcal{C}_{\mathcal{A}_1}(\Gamma) = \{\Lambda | B(\mathcal{A}_1) \mid \Lambda \in \mathcal{C}_{\mathcal{A}_2}(\Gamma)\}$.

Proof.

(a) Let $\Lambda \in \mathcal{C}_{\mathcal{A}_1}(\Gamma)$ and $f \in B(\mathcal{A}_1)$. Then using exactness of Λ and $|\Lambda| = \Lambda(1)$, by Proposition 1.2.4,

$$\begin{aligned}
\Gamma_\bullet(f) &= \sup\left\{ \sum_{i=1}^n \lambda_i \Gamma(f_i) + \lambda_0 |\Gamma| \;\middle|\; \sum_{i=1}^n \lambda_i f_i + \lambda_0 \le f \right\} \\
&\le \sup\left\{ \sum_{i=1}^n \lambda_i \Lambda(f_i) + \lambda_0 |\Lambda| \;\middle|\; \sum_{i=1}^n \lambda_i f_i + \lambda_0 \le f \right\} \\
&= \sup\left\{ \Lambda\left(\sum_{i=1}^n \lambda_i f_i + \lambda_0 \right) \;\middle|\; \sum_{i=1}^n \lambda_i f_i + \lambda_0 \le f \right\} \\
&= \Lambda(f),
\end{aligned}$$

thus Λ dominates the superlinear functional $\Gamma_\bullet | B(\mathcal{A}_1)$. By the Hahn-Banach Theorem, Λ can be extended to a linear functional Λ' on $B(2^\Omega)$ such that $|\Lambda'| = |\Lambda|$ and $\Lambda' \ge \Gamma_\bullet$. Λ' is monotone since $f \ge 0$ implies $\Lambda'(f) \ge \Gamma_\bullet(f) \ge 0$, i.e. $\Lambda' \in \mathcal{C}_{2^\Omega}(\Gamma_\bullet)$. Thus we have $\mathcal{C}_{\mathcal{A}_1}(\Gamma) \subset \{\Lambda | B(\mathcal{A}_1) \mid \Lambda \in \mathcal{C}_{2^\Omega}(\Gamma_\bullet)\}$. The reverse inclusion is trivial.

(b) The assertion follows from (a) by replacing $\mathcal{C}_{\mathcal{A}_2}$ in (b) by $\{\Lambda | B(\mathcal{A}_2) \mid \Lambda \in \mathcal{C}_{2^\Omega}(\Gamma_\bullet)\}$. □

Proposition 1.5.5 (a) allows us to investigate the core of an exactifiable functional by investigating its natural exactification. Therefore we can restrict our examinations to the relations between exact functionals on $B(2^\Omega)$ and their core and obtain afterwards the general results just by applying Proposition 1.5.5 (a). These examinations start with one further representation of the core of an exact functional $\Gamma : B(2^\Omega) \to \mathbb{R}$ taking advantage of the structure of the domain of Γ. We will make use of the natural embedding $B(2^\Omega) \to B^{**}(2^\Omega)$, $f \mapsto \hat{f}$ defined by $\hat{f}(\Lambda) := \Lambda(f)$.

Lemma 1.5.6 *Let* Γ *be an exact functional on* $B(2^\Omega)$. *Then*

$$\mathcal{C}(\Gamma) = \left\{\Lambda \in B^*(2^\Omega) \mid \Gamma \leq \Lambda \leq \overline{\Gamma}\right\} \tag{1.31}$$

$$= \bigcap_{f \in B(2^\Omega)} \hat{f}^{-1}\big([\Gamma(f), \overline{\Gamma}(f)]\big). \tag{1.32}$$

Proof. Suppose $\Lambda \in \mathcal{C}(\Gamma)$. Then $\Lambda(f) = -\Lambda(-f) \leq -\Gamma(-f) = \overline{\Gamma}(f)$, hence $\Lambda \in \{\Lambda' \in B^*(2^\Omega) \mid \Gamma \leq \Lambda' \leq \overline{\Gamma}\}$.
Now let $\Lambda : B(2^\Omega) \to \mathbb{R}$ satisfy $\Gamma \leq \Lambda \leq \overline{\Gamma}$. Then Λ is monotone since $f \geq 0$ implies $\Lambda(f) \geq \Gamma(f) \geq 0$. By Proposition 1.2.2, $\Gamma(1) = \overline{\Gamma}(1)$, hence $\Lambda(1) = \Gamma(1)$ and using Proposition 1.2.4

$$|\Lambda| = \Lambda(1) = \Gamma(1) = |\Gamma|.$$

Equation (1.32) holds by the definition of the natural embedding. □

The subsequent theorem is essential to adopt results from (σ-)additive measure and integration theory to the theory of exact functionals like for example convergence theorems (cf. Theorem 1.5.11). It has been proved for superlinear functionals by Bonsall (Bonsall 1954 [5, Lemma 6 and Theorem 11]) and later independently in different contexts, e.g. by Huber (Huber 1981 [23, Proposition 10.2.1]. The main part of our proof is a simple application of the Hahn-Banach Theorem.

Theorem 1.5.7 *There is a one-to-one correspondence between exact functionals on* $B(2^\Omega)$ *and non-empty, convex, weak*-compact sets* $\mathcal{C} \subset B^*(2^\Omega)$ *of equinormed functionals, determined by the identities*

$$\Gamma(f) = \min_{\Lambda \in \mathcal{C}} \Lambda(f) \quad \textit{resp.} \quad \mathcal{C} = \mathcal{C}(\Gamma). \tag{1.33}$$

Proof. First, we prove that $\min_{\Lambda \in \mathcal{C}} \Lambda$ is exact. For every $f \in B(2^\Omega)$, the natural embedding $\hat{f}$ attains its infimum on $\mathcal{C}$ because of the weak*-compactness of $\mathcal{C}$. Exactness of $\min_{\Lambda \in \mathcal{C}} \Lambda$ then follows from Corollary 1.4.4.
Next, $\mathcal{C}_1 \neq \mathcal{C}_2$ implies $\min_{\Lambda \in \mathcal{C}_1} \Lambda \neq \min_{\Lambda \in \mathcal{C}_2} \Lambda$. Let w.l.o.g. $\Lambda' \in \mathcal{C}_1 \setminus \mathcal{C}_2$. Then by a separation theorem (Dunford and Schwartz 1958 [14, Theorem V.2.10]) there exists a $f \in B(2^\Omega)$ with $\Lambda'(f) < \min_{\Lambda \in \mathcal{C}_2} \Lambda(f)$, thus $\min_{\Lambda \in \mathcal{C}_1} \Lambda \neq \min_{\Lambda \in \mathcal{C}_2} \Lambda$.
Now we prove that $\mathcal{C}(\Gamma)$ is non-empty, convex and weak*-compact and

$\Gamma = \min_{\Lambda \in \mathcal{C}(\Gamma)} \Lambda$. Convexity of $\mathcal{C}(\Gamma)$ follows directly from Definition 1.5.1. Equation (1.32) in Lemma 1.5.6 implies weak*-closeness of $\mathcal{C}(\Gamma)$. If $|\Gamma| = 0$ then $\mathcal{C}(\Gamma) = \{0\}$, i.e. the core is weak*-compact. If $|\Gamma| > 0$ then the core is weak*-compact as a subset of a positive multiple of the weak*-closed unit ball in $B^*(2^\Omega)$, which is weak*-compact by the Banach-Alaoglu Theorem (cf. Dunford and Schwartz 1958 [14, Theorem V.4.2]).
To prove non-emptiness of the core and $\Gamma = \min_{\Lambda \in \mathcal{C}(\Gamma)} \Lambda$ we show that for every $f_0 \in B(2^\Omega)$ there exists a $\Lambda \in B^*(2^\Omega)$ with $\Lambda(1) = \Gamma(1)$, $\Lambda(f_0) = \Gamma(f_0)$ and $\Lambda(f) \geq \Gamma(f)$ for all $f \in B(2^\Omega)$. Let $f_0 \in B(2^\Omega)$ be arbitrary and the linear functional Λ' on the linear space spanned by the functions 1 and f_0, $\text{span}(1, f_0)$, be defined by $\Lambda'(1) := \Gamma(1)$ and $\Lambda'(f_0) := \Gamma(f_0)$. Then $\Lambda' \geq \Gamma|\text{span}(1, f_0)$ because of $\Lambda'(-f_0) = -\Lambda'(f_0) = -\Gamma(f_0) \geq \Gamma(-f_0)$. Using the Hahn-Banach Theorem we can extend Λ' to $B(2^\Omega)$ such that this extension is contained in $\mathcal{C}(\Gamma)$.
Finally, $\Gamma_1 \neq \Gamma_2$ implies $\mathcal{C}(\Gamma_1) \neq \mathcal{C}(\Gamma_2)$. If $\Gamma_1 \neq \Gamma_2$ there is w.l.o.g. a $f \in B(2^\Omega)$ with $\Gamma_1(f) < \Gamma_2(f)$. Then by the preceding part of the proof there exists a $\Lambda \in \mathcal{C}(\Gamma_1)$ with $\Lambda(f) = \Gamma(f)$. Therefore $\mathcal{C}(\Gamma_1) \neq \mathcal{C}(\Gamma_2)$ because of $\Lambda \notin \mathcal{C}(\Gamma_2)$. □

The next corollary characterizes the natural extension, resp. the natural exactification, by means of the core in an analogous way as in Proposition 1.2.7. Corollary 1.5.8 (a) is well-known in game theory in a slightly different form (cf. Schmeidler 1972 [35, Corollary 2.6]).

Corollary 1.5.8 *Let $\Gamma : M \to \mathbb{R}$ be a functional.*

(a) If Γ is exact then for all $f \in B(2^\Omega)$

$$\Gamma_* = \min\big\{\Lambda \mid \Lambda \in B^*(2^\Omega), \Lambda|M \geq \Gamma, \Lambda \text{ monotone}, \|\Lambda\| = \|\Gamma\|\big\}. \tag{1.34}$$

(b) If Γ is exactifiable then for all $f \in B(2^\Omega)$

$$\Gamma_\bullet = \min\big\{\Lambda \mid \Lambda \in \mathcal{C}_{2^\Omega}(\Gamma)\big\}. \tag{1.35}$$

Proof.

(a) Using $\|\Gamma\| = \|\Gamma_*\|$ by Theorem 1.2.5 and $|\Lambda| = \|\Lambda\|$ by Proposition 1.2.4, we obtain from Theorem 1.5.7

$$\mathcal{C}_{2^\Omega}(\Gamma_*) = \big\{\Lambda \in B^*(2^\Omega) \ \mid \ \Lambda \geq \Gamma_*, \Lambda \text{ monotone}, \|\Lambda\| = \|\Gamma\|\big\}.$$

Let $\Lambda \in \{\Lambda' \in B^*(2^\Omega) \mid \Lambda'|M \geq \Gamma, \Lambda'$ monotone, $\|\Lambda'\|$

Definition 1.5.10 *Let $\Gamma : M \to \mathbb{R}$ be a functional, $\mathcal{A}_M$ the σ-algebra generated by the upper level sets of all $f \in M$, i.e. $\mathcal{A}_M := \mathcal{A}(\{f \geq \alpha\},\ f \in M, \alpha \in \mathbb{R})$. Then*

$$\mathcal{C}^\sigma(\Gamma) := \{\lambda \in ca(\mathcal{A}_M) \mid I_\lambda|M \geq \Gamma, \lambda \geq 0, \lambda(\Omega) = |\Gamma|\} \tag{1.36}$$

*is called the **σ-core** of Γ.*

Comparing the definitions of the core and the σ-core, a unique "small" σ-algebra $\mathcal{A}_M$ is used in the latter. This is done due to the fact that in contrast to the elements of the core those of the σ-core cannot be extended to σ-additive measures on greater domains in general, i.e. Proposition 1.5.5 does not hold for the σ-core (cf. Parker 1991 [33, Example 1]).

Due to the close connection between exact functionals and their core the continuity properties of exact functionals correspond directly to those of the elements of the $\mathcal{A}_M$-core like in game theory (cf. Schmeidler 1972 [35, Theorem 3.2 and Proposition 3.15]). The proof of the subsequent theorem is a simple generalization of that given by Parker for exact games in (Parker 1991 [33]).

Theorem 1.5.11 (Monotone Convergence Theorem)
Let $\Gamma : M \to \mathbb{R}$ be an exact functional satisfying $\mathcal{C}(\Gamma) = \mathcal{C}^\sigma(\Gamma)$ and $(f_n)_{n\in\mathbb{N}}$ a monotone sequence in M such that f_n converges pointwise to a function $f \in M$. Then

$$\lim_{n\to\infty} \Gamma(f_n) = \Gamma(f). \tag{1.37}$$

Proof. Suppose $(f_n)_{n\in\mathbb{N}}$ is a decreasing sequence in M such that $\lim_{n\to\infty} f_n = f \in M$. Since Γ is exact, there exists an element Λ in the core $\mathcal{C}(\Gamma)$ by Proposition 1.5.7 such that $\Lambda(f) = \Gamma(f)$ and since $\mathcal{C}(\Gamma) = \mathcal{C}^\sigma(\Gamma)$,

$$\Gamma(f) \leq \inf_{n\in\mathbb{N}} \Gamma(f_n) \leq \inf_{n\in\mathbb{N}} \Lambda(f_n) = \Lambda(f) = \Gamma(f).$$

Now suppose $(f_n)_{n\in\mathbb{N}}$ is an increasing sequence in M such that $\lim_{n\to\infty} f_n = f \in M$. Then $\hat{f}_n : \mathcal{C}_{\mathcal{A}_M}(\Gamma) \to \mathbb{R}$ is an increasing sequence of weak*-continuous functions on $\mathcal{C}_{\mathcal{A}_M}(\Gamma)$ with pointwise limit $\hat{f}$ because $\mathcal{C}(\Gamma) = \mathcal{C}^\sigma(\Gamma)$. By Dini's Theorem (cf. Dudley 1989 [13, Theorem 2.4.10]) the convergence is uniform,

$$\Gamma(f) = \min_{\Lambda\in\mathcal{C}(\Gamma)} \Lambda(f) = \min_{\Lambda\in\mathcal{C}(\Gamma)} \sup_{n\in\mathbb{N}} \Lambda(f_n) = \sup_{n\in\mathbb{N}} \min_{\Lambda\in\mathcal{C}(\Gamma)} \Lambda(f_n) = \sup_{n\in\mathbb{N}} \Gamma(f_n).$$

$\square$

The reverse implication of the Monotone Convergence Theorem, i.e. the deduction of $\mathcal{C}(\Gamma) = \mathcal{C}^{\sigma}(\Gamma)$ from the continuity property (1.37) is not valid in general. This is based on the fact that the domain of Γ can have not enough structure to relate the equivalence of the cores to the continuity property of Γ.

Fatou's Lemma and Lebesgue's Dominated Convergence Theorem can be deduced from the Monotone Convergence Theorem analogously to integration theory.

Chapter 2

Linear Representation of Linear Inequality Preserving Functionals

2.1 Introduction

In different fields of mathematical economics, e.g. game theory or decision theory, one naturally arrives at non-additive set functions and non-linear functionals. For these classes of functions, there is not such an elaborated mathematical theory like for measures or linear functionals and the existing results have not become very popular among most mathematicians. Since in most cases in mathematical economics one starts with a linear space of non-additive set functions or non-linear functionals and singles out a special subclass within this space it would be favorable to have a transform at our disposal mapping each element of the linear space to a signed measure or linear functional and characterize the elements of the special subclass by monotonicity of the transformed function.

For totally monotone set functions, such a result is already known as Dempster-Shafer-Shapley Representation Theorem in the discrete case or as Möbius transform in the general case. The results in this chapter mainly base on articles from Glenn Shafer (cf. Shafer 1979 [38]), Massimo Marinacci (cf. Marinacci [32]), and Dieter Denneberg (cf. Denneberg 1997 [12]). We pick up an idea first stated by Shafer in 1979 using Choquet's Theorem for introducing the transform. This method allows us not to presuppose special properties of the domain of the considered set functions, resp. functionals. We provide a Möbius type transform for several classes of non-additive set functions, resp. non-linear functionals, principally using one property shared by all of them – preserving certain linear inequalities.

This chapter is organized as follows. In Section 2, we state the Bishop-de Leeuw Theorem which, like Choquet's Theorem, belongs to a group of results generalizing the famous Krein-Milman Theorem and we provide some topological foundations necessary for its application. The notion of linear inequality preserving functionals is introduced in Section 3. We point out the generality of this notion by giving several examples and we state some elementary topological results. In Section 4, we present the announced isomorphism between the linear space spanned by the linear inequality preserving functionals and a linear space of restricted integrals and characterize the linear inequality preserving functionals by monotonicity of their transformed.

2.2 Preliminaries

Like in the previous chapter, let Ω be a non-empty set, $B(2^\Omega)$ the linear space of bounded (w.r.t. the supremum norm) real-valued functions on Ω and $M \subset B(2^\Omega)$ be non-empty. To avoid laborious considerations of special cases, we will assume the constant function 1 to be contained in M. If M consists of all indicator functions of the elements of an algebra then a functional Γ on M can be interpreted as a set function. In this case, $1 \in M$ is trivially fulfilled since every algebra contains the whole set Ω. Denote by $B(M)$ the linear space of all real-valued functionals Γ on M which are bounded, i.e. the operator norm $\|\cdot\|_{\mathrm{op}} : M \to \mathbb{R}$, $\|\Gamma\|_{\mathrm{op}} := \sup_{f\in M, f\neq 0} \frac{|\Gamma(f)|}{\|f\|_\infty}$ is finite.

The linear space $B(M)$ will additionally be considered as a topological space endowed with the topology $\mathcal{T}$ having as subbase the sets $B(\Gamma, f, \varepsilon) := \{\Gamma' \in B(M) \mid |\Gamma'(f) - \Gamma(f)| < \varepsilon\}$, with $\Gamma \in B(M)$, $f \in M$, and $\varepsilon > 0$. The definition of $\mathcal{T}$ is similar to that of the weak* topology and it is the smallest making all functions

$$\tilde{f} : B(M) \to \mathbb{R}, \quad \tilde{f}(\Gamma) := \Gamma(f) \tag{2.1}$$

continuous for all $f \in M$. The set of all such $\tilde{f}$ will be denoted by $\tilde{M}$, the linear space spanned by $\tilde{M}$ will be denoted by $\mathrm{span}(\tilde{M})$. The topology $\mathcal{T}$ is also known as the topology of pointwise convergence and, by definition of the product topology, $\mathcal{T}$ is identical with the relative topology of $B(M)$ as a subset of the product space $\Pi_{f\in M}\mathbb{R}_f$, $\mathbb{R}_f := \mathbb{R}$ for all $f \in M$.

We start with some topological results that will serve as technical basis for the following analysis.

Proposition 2.2.1 *Under the topology $\mathcal{T}$ the linear space $B(M)$ is a locally convex and Hausdorff topological linear space.*

Proof. We have to show that $\mathcal{T}$ possesses a base consisting of convex sets. Since convexity is preserved under forming intersections it suffices to show that the given subbase of $\mathcal{T}$ consists of convex sets. Therefore, suppose $\Gamma_1, \Gamma_2 \in B(\Gamma, f, \varepsilon)$ with $\Gamma \in B(M)$, $f \in M$ and $\varepsilon > 0$ and let $\lambda \in [0,1]$. Then

$$|\lambda\Gamma_1(f)+(1-\lambda)\Gamma_2(f)-\Gamma(f)| \leq \lambda|\Gamma_1(f)-\Gamma(f)|+(1-\lambda)|\Gamma_2(f)-\Gamma(f)| < \varepsilon,$$

i.e. $B(\Gamma, f, \varepsilon)$ is convex since Γ_1, Γ_2 and λ were chosen arbitrarily. Therefore, all elements of the subbase are convex because Γ, f and ε were chosen arbitrarily. □

The subsequent proposition is a variant of the Banach-Alaoglu Theorem which states that the closed unit ball is weak*-compact (cf. Dunford and Schwartz 1958 [14, Theorem V.4.2]) .

Proposition 2.2.2 *The unit ball $B_1 = \{\Gamma \in B(M) \mid \|\Gamma\|_{\mathrm{op}} \leq 1\}$ in $(B(M), \|\cdot\|_{\mathrm{op}})$ is $\mathcal{T}$-compact.*

Proof. Let $I := \Pi_{f\in M}[-1,1]$. By Tychonoff's Theorem, I is compact w.r.t. the product topology. Let $\tau : B_1 \to I$ be the injective mapping $\tau(\Gamma) := \Pi_{f\in M}\frac{\Gamma(f)}{\|f\|_\infty}$. Since the sets $B(\Gamma, f, \varepsilon) := \{\Gamma' \in B_1 \mid |\Gamma'(f) - \Gamma(f)| < \varepsilon\}$ with $\Gamma \in B_1$, $f \in M$, $\varepsilon > 0$ form a subbase for the relative topology $\mathcal{T}_B$ of B_1 generated by $\mathcal{T}$ and since $\{\Pi_{f\in M}U_f \mid \exists f' \in M, x \in \mathbb{R}, \varepsilon > 0 : U_f = \mathbb{R}\ \forall f \in M \setminus \{f'\}, U_{f'} =]x-\varepsilon, x+\varepsilon[\}$ is a subbase of the product topology in $\mathbb{R}^M$, the images of the subbase elements of $\mathcal{T}_{B_1}$ form a subbase of the relative product topology in $\tau(B_1)$. Thus τ is a homeomorphism between B_1 endowed with the relative $\mathcal{T}$-topology, and $\tau(B_1)$ endowed with the relative product topology. Therefore, to prove that B_1 is $\mathcal{T}$-compact, it suffices to show that B_1 is $\mathcal{T}$-closed. This is easily done since for any $\Gamma \in B(M)$ with

$\|\Gamma\|_{\mathrm{op}} > 1$ there exist an $f \in M$ and an $\varepsilon > 0$ with $|\Gamma(f)| > \|f\|_\infty + \varepsilon$ such that $B(\Gamma, f, \varepsilon)$ is an open neighborhood of Γ disjoint from B_1, i.e. B_1 is $\mathcal{T}$-closed. □

The main result of this chapter will heavily base on the Bishop-de Leeuw Theorem (cf. Alfsen 1971 [1, Theorem I.4.14]). We recall that the Baire σ-algebra of a topological space is the smallest σ-algebra for which all continuous real-valued functions are measurable, with, as usual, the Borel σ-algebra on the range space $\mathbb{R}$. Furthermore, denote by $\mathrm{ex}(X)$ the set of extreme points of X.

Theorem 2.2.3 (Bishop-de Leeuw) *Suppose E is a locally convex Hausdorff linear space over $\mathbb{R}$ and X a non-empty compact convex subset of E. Denote by $A(X)$ the linear space of continuous real-valued functions $a : X \to \mathbb{R}$ which are affine, i.e. $a(\lambda x + (1-\lambda)y) = \lambda a(x) + (1-\lambda)a(y)$ for $x, y \in X$, $0 \le \lambda \le 1$ and by $\mathcal{B}_0$ the Baire σ-algebra on X. Then for every $x \in X$ there exists a probability measure μ_x on the σ-algebra $\mathrm{ex}(X) \cap \mathcal{B}_0$, such that*

$$a(x) = \int a \, d\mu_x \quad \textit{for all } a \in A(X). \tag{2.2}$$

Generally, it is not possible to replace the Baire σ-algebra by the greater Borel σ-algebra (cf. Alfsen 1971 [1, p. 39 f.]).

2.3 Linear inequality preserving functionals

Throughout this chapter, denote by $\mathcal{S}$ system of finite sets in $\mathbb{R} \times M$ satisfying

$$\{(x_i, f_i) \mid i = 1, \ldots, n\} =: (x_i, f_i)_{i=1,\ldots,n} \in \mathcal{S} \quad \Longrightarrow \quad \sum_{i=1}^{n} x_i f_i \ge 0. \tag{2.3}$$

Furthermore, let $C(\mathcal{S})$ denote the class of *linear inequality preserving functionals* $\Gamma \in B(M)$ (w.r.t. $\mathcal{S}$) determined by

$$\Gamma \in C(\mathcal{S}) \quad :\Longleftrightarrow \quad \sum_{i=1}^{n} x_i \Gamma(f_i) \ge 0 \quad \forall\, (x_i, f_i)_{i=1,\ldots,n} \in \mathcal{S}. \tag{2.4}$$

Subsequently, we provide some examples of linear inequality preserving functionals. In fact, most classes of non-additive set functions and non-linear functionals occurring in mathematical economics are linear inequality preserving functionals.

Example 2.3.1

(a) $C(\emptyset) = B(M)$. *Moreover,* $C(\mathcal{S}) = B(M)$ *if* $(x_i, f_i)_{i=1,\ldots,n} \in \mathcal{S}$ *implies* $x_i = 0$ *for all* $i = 1, \ldots, n$.

(b) Linear functionals are the linear inequality preserving functionals w.r.t.

$$\mathcal{S}_{lin} := \Big\{ (x_i, f_i)_{i=1,\ldots,n} \;\Big|\; n \in \mathbb{N}, \sum_{i=1}^{n} x_i f_i = 0 \Big\}. \tag{2.5}$$

Linearity of all $\Gamma \in C(\mathcal{S}_{lin})$ *follows from the fact that* $(x_i, f_i)_{i=1,\ldots,n} \in \mathcal{S}_{lin}$ *implies* $(-x_i, f_i)_{i=1,\ldots,n} \in \mathcal{S}_{lin}$.

(c) Monotone functionals are the linear inequality preserving functionals w.r.t.

$$\mathcal{S}_{mon} := \Big\{ \{(1, f), (-1, g)\} \;\Big|\; f \geq g \Big\}. \tag{2.6}$$

(d) Positive linear functionals are the linear inequality preserving functionals w.r.t.

$$\mathcal{S}_{lin} \cup \mathcal{S}_{mon}. \tag{2.7}$$

(e) For a natural number $k \geq 2$, *a cooperative game* v *on an algebra* $M = \mathcal{A}$ *is called* k*-monotone if for any* $A_1, \ldots, A_k \in \mathcal{A}$

$$v\Big(\bigcup_{i=1}^{k} A_i\Big) + \sum_{I \subset \{1,\ldots,k\}, I \neq \emptyset} (-1)^{|I|} \, v\Big(\bigcap_{i \in I} A_i\Big) \geq 0. \tag{2.8}$$

Hence k*-monotone games are the linear inequality preserving set functions w.r.t.*

$$\mathcal{S}_{k-mon} \;:=\; \Big\{ \Big((-1)^{|I|}, \bigcap_{i \in I} A_i \Big)_{I \subset \{1,\ldots,k\}} \;\Big|\; A_1, \ldots, A_n \in \mathcal{A} \Big\} \tag{2.9}$$

understanding that $\bigcap_{i \in \emptyset} A_i = \bigcup_{i=1}^{k} A_i$. *The elements of* $\mathcal{S}_{k-mon}$ *fulfill Condition (2.3) even with equality by the principle of inclusion exclusion.*

(f) *A cooperative game is called totally monotone if it is monotone and k-monotone, for all $k \geq 2$. Thus, totally monotone games are the linear inequality preserving functionals w.r.t.*

$$\mathcal{S}_{tot-mon} = \mathcal{S}_{mon} \cup \bigcup_{k \in \mathbb{N}} \mathcal{S}_{k-mon}. \tag{2.10}$$

(g) *Exact functionals (recall that we presuppose $1 \in M$) are the linear inequality preserving functionals w.r.t.*

$$\begin{aligned} \mathcal{S}_{exact} \quad := \quad & \Big\{ (x_i, f_i)_{i=1,\ldots,n+2} \mid x_1, \ldots, x_n < 0, x_{n+1} \in \mathbb{R}, \\ & \qquad x_{n+2} = 1, f_{n+1} = 1, \sum_{i=1}^{n+2} x_i f_i \geq 0 \Big\}. \end{aligned} \tag{2.11}$$

This can be rewritten to the usual definition of exact functionals,

$$f_{n+2} \geq \sum_{i=1}^{n} x_i f_i + x_{n+1} \implies \Gamma(f_{n+2}) \geq \sum_{i=1}^{n} x_i \Gamma(f_i) + x_{n+1}\Gamma(1) \tag{2.12}$$

for all $n \in \mathbb{N}$, $x_1, \ldots, x_n > 0$, $x_{n+1} \in \mathbb{R}$.

Now we state some basic properties of linear inequality preserving functionals.

Proposition 2.3.2 *$C(\mathcal{S})$ is a $\mathcal{T}$-closed convex cone.*

Proof. It follows directly from the definition that $C(\mathcal{S})$ is a convex cone. To prove that $C(\mathcal{S})$ is $\mathcal{T}$-closed suppose $\Gamma \notin C(\mathcal{S})$. Then there exists a finite sequence $(x_i, f_i)_{i=1,\ldots,n}$ in $\mathcal{S}$ with $\sum_{i=1}^{n} x_i \Gamma(f_i) < 0$. Setting $\varepsilon_i := \varepsilon / (2 \sum_{k=1}^{n} x_k)$, the set $\bigcap_{i=1}^{n} B(\Gamma, f_i, \varepsilon_i)$ is an open neighborhood of Γ which is disjoint from $C(\mathcal{S})$. Hence $C(\mathcal{S})$ is $\mathcal{T}$-closed. □

The subsequent proposition follows directly from Proposition 2.2.2 and Proposition 2.3.2.

Proposition 2.3.3 *The set $C(\mathcal{S}) \cap B_1$ is $\mathcal{T}$-compact in $B(M)$.*

Note, that $C(\mathcal{S}_{tot-mon}) \cap B_1$ consists of all belief functions and that $C(\mathcal{S}_{exact}) \cap B_1$ consists of all coherent lower previsions (cf. Proposition 1.3.5 and Proposition 1.2.4).

We now show that $C(\mathcal{S})$ is not only closed under (finite) convex combinations but also under "infinite convex combinations" (expected values). The subsequent proposition is very similar to Theorem 1.4.3.

Proposition 2.3.4 *Let $X \subset C(\mathcal{S})$ and $\Lambda : \{\tilde{f}|_X \mid f \in M\} \to \mathbb{R}$ be a monotone linear functional. Then the functional $\Gamma : M \to \mathbb{R}$, defined by*

$$\Gamma(f) := \Lambda(\tilde{f}) \tag{2.13}$$

is contained in $C(\mathcal{S})$. If especially $\mathcal{A}$ is an arbitrary σ-algebra over X making all $\tilde{f}|_X$ measurable, $f \in M$, and μ is a probability measure on $\mathcal{A}$ then the functional $\Gamma : M \to \mathbb{R}$, defined by

$$\Gamma(f) := \int \tilde{f}\, d\mu \tag{2.14}$$

is contained in $C(\mathcal{S})$.

Proof. Let $(x_i, f_i)_{i=1,\dots,n}$ be in $\mathcal{S}$. From $\sum_{i=1}^n x_i \Gamma'(f_i) \geq 0$ for every $\Gamma' \in X$ follows $\sum_{i=1}^n x_i \tilde{f}_i \geq 0$. Thus

$$\sum_{i=1}^n x_i \Gamma(f_i) = \sum_{i=1}^n x_i \Lambda(\tilde{f}_i) = \Lambda\Big(\sum_{i=1}^n x_i \tilde{f}_i\Big) \geq 0. \qquad \square$$

2.4 Main results

In this section, we present the announced isomorphism between the linear space spanned by a convex cone of linear inequality preserving functionals $C(\mathcal{S})$ and a linear space of restricted integrals and characterize the elements of $C(\mathcal{S})$ by monotonicity of their transform. As a preparation, we start with a simple application of the Bishop-de Leeuw Theorem.

Lemma 2.4.1 *For every linear inequality preserving functional $\Gamma \in C(\mathcal{S}) \cap B_1$ on M, there exists a measure $\mu_\Gamma : \mathrm{ex}(C(\mathcal{S}) \cap B_1) \cap \mathcal{B}_0 \to \mathbb{R}_+$, such that*

$$\Gamma(f) = \int \tilde{f}\, d\mu_\Gamma \quad \text{for all } f \in M. \tag{2.15}$$

Proof. The assertion follows directly from Theorem 2.2.3 using Proposition 2.2.1 and Proposition 2.3.3 and from $\tilde{f} \in A(C(\mathcal{S}))$ for all $f \in M$. □

We obviously have found that the continuous linear functional $\int \cdot \, d\mu_\Gamma$ represents the linear inequality preserving functional Γ via the nonlinear application $f \mapsto \tilde{f}$. For totally monotone games (i.e. belief functions in the normalized case), it is well-known that the transform (which is called Möbius transform) is unique. For supermodular (i.e. 2-monotone) and exact set functions, the representing measure μ_Γ needs not to be unique as the following example shows.

Example 2.4.2 *Let $\Omega = \{1,2,3\}$ and $\nu : 2^\Omega \to \mathbb{R}$ be the normalized supermodular (hence exact) set function defined by $\nu(A) := \frac{1}{2}$ if and only if $|A| = 2$ and $\nu(A) := 0$ if and only if $|A| < 2$. Then ν is an extreme point*[1] *of the set of normalized supermodular, resp. exact, set functions on 2^Ω. Suppose ν is a convex combination of two supermodular, resp. exact, set functions ν_1 and ν_2. Obviously, $\nu_1(A) = \nu_2(A) = \nu(A)$ for all A with $\nu(A) \in \{0,1\}$, i.e. $|A| \neq 2$. Therefore, suppose $\nu_1(\{1,2\}) > \nu(\{1,2\}) = \frac{1}{2}$. By supermodularity, resp. exactness, of ν_1, $1_{\{1\}} \geq 1_{\{1,2\}} + 1_{\{1,3\}} - 1$ implies $\nu_1(\{1\}) \geq \nu_1(\{1,2\}) + \nu_1(\{1,3\}) - 1$ such that $\nu_1(\{1,3\}) < \frac{1}{2}$. Analogously, we conclude $\nu_1(\{2,3\}) < \frac{1}{2}$. The same argument applied to ν_2 implies that both ν_1 and ν_2 are at least for two of three sets A with $|A| = 2$ smaller than or equal to $\nu(A)$. Hence $\nu_1 = \nu_2 = \nu$.*
Further on, it is easy to see that all unanimity games on 2^Ω are extreme points of the set of normalized supermodular, resp. exact, set functions.
The normalized supermodular (hence exact) set function $\nu' : 2^\Omega \to \mathbb{R}$ defined by $\nu'(A) := \frac{1}{3}$ if and only if $|A| = 2$ and $\nu'(A) := 0$ if and only if $|A| < 2$ can be obtained by two different convex combinations of extreme points, $\nu' = \frac{1}{3}u_{\{1,2\}} + \frac{1}{3}u_{\{1,3\}} + \frac{1}{3}u_{\{2,3\}}$ and $\nu' = \frac{2}{3}\nu + \frac{1}{3}u_\Omega$. Since the coefficients of the extreme points used in the convex combinations are the masses of the transform $\mu_{\nu'}$ of ν', we obtain that uniqueness of the representing measure cannot be guaranteed.

To obtain uniqueness independently from the considered class of linear inequality preserving functionals, we have to draw our attention to the integrals because for two representing measures μ_Γ and μ'_Γ of Γ we have, by

[1] It can be shown that ν is the only non-unanimity game in the set of extreme points of the set of normalized supermodular, resp. exact, set functions.

Lemma 2.4.1,

$$\int \tilde{f}\, d\mu_\Gamma = \int \tilde{f}\, d\mu'_\Gamma \quad \text{for all } f \in M. \tag{2.16}$$

So, if we just restrict the continuous linear functional $\int \cdot\, d\mu_\Gamma$ to the linear space $\text{span}(\tilde{M})$ we get the desired uniqueness.

By merely collecting the results from Lemma 2.4.1, the remarks in the preceding paragraph (especially Equation (2.16)) and Proposition 2.3.4, we obtain the subsequent proposition which contains the essential mathematical part of the main theorem of this chapter (Theorem 2.4.4).

Proposition 2.4.3 *The mapping*

$$\begin{aligned} C(\mathcal{S}) &\to \Big\{ \big(\int \cdot\, d\mu\big)\big|_{\text{span}(\tilde{M})} \,\Big|\, \mu : \text{ex}(C(\mathcal{S}) \cap B_1) \cap \mathcal{B}_0 \to \mathbb{R}_+ \textit{ measure} \Big\}, \\ \Gamma &\mapsto \big(\int \cdot\, d\mu_\Gamma\big)\big|_{\text{span}(\tilde{M})} \end{aligned} \tag{2.17}$$

with $\Gamma(f) = \int \tilde{f}\, d\mu_\Gamma$ *for all* $f \in M$ *is bijective.*

We now expand this first result to the linear spaces spanned by the respective sets used in Proposition 2.4.3. Thus, denote by

$$V_1 := \Big\{ \Gamma_1 - \Gamma_2 \,\Big|\, \Gamma_1, \Gamma_2 \in C(\mathcal{S}) \Big\} \tag{2.18}$$

the linear space of functionals spanned by $C(\mathcal{S})$ and by

$$\begin{aligned} V_2 \;:=\; \Big\{ \big(\int \cdot\, d\mu\big)\big|_{\text{span}(\tilde{M})} \,\Big|\, &\mu : \text{ex}(C(\mathcal{S}) \cap B_1) \cap \mathcal{B}_0 \to \mathbb{R} \\ &\text{measure of bounded variation} \Big\}. \end{aligned} \tag{2.19}$$

the linear space of restricted integrals w.r.t. signed measures on $\text{ex}(C(\mathcal{S}) \cap B_1) \cap \mathcal{B}_0$ of bounded variation. Let V_1 be endowed with $\mathcal{T}_{V_1}$, the relative topology of V_1 generated by $\mathcal{T}$, i.e. the smallest topology making all $\tilde{f}$ restricted to V_1 continuous and let V_2 be endowed with $\mathcal{T}_{V_2}$, the weak* topology, i.e. the smallest topology making all natural embeddings $\tilde{\tilde{f}} : V_2 \to \mathbb{R}$, $\tilde{\tilde{f}}\Big(\big(\int \cdot\, d\mu\big)\big|_{\text{span}(\tilde{M})} \Big) := \int \tilde{f}\, d\mu$, continuous.

Theorem 2.4.4 *There is a linear isomorphism $\varphi : V_1 \to V_2$ between the linear spaces defined in (2.18) and (2.19). The isomorphism is determined by the identity*

$$\Gamma(f) = \int \tilde{f}\, d\mu \quad \text{for all } f \in M. \tag{2.20}$$

The isomorphism φ is topological, i.e. a homeomorphism, between the topological spaces $(V_1, \mathcal{T}_{V_1})$ and $(V_2, \mathcal{T}_{V_2})$. Moreover, Γ is contained in $C(\mathcal{S})$ if and only if its transformed $\varphi(\Gamma)$ is monotone.

Proof. To prove that φ is well defined, it suffices to show that for every $\Gamma \in V_1$ there is a measure $\mu : \mathrm{ex}(C(\mathcal{S}) \cap B_1) \cap \mathcal{B}_0 \to \mathbb{R}$ of bounded variation with $\Gamma(f) = \int \tilde{f}\, d\mu$ for all $f \in M$ because uniqueness of the image is guaranteed by Equation (2.20). Suppose $\Gamma = \Gamma_1 - \Gamma_2$ with $\Gamma_1, \Gamma_2 \in C(\mathcal{S})$. Then, by Proposition 2.4.3, there exist measures μ_1, μ_2 on $\mathrm{ex}(C(\mathcal{S}) \cap B_1) \cap \mathcal{B}_0$ satisfying $\Gamma_1(f) = \int \tilde{f}\, d\mu_1$ and $\Gamma_2(f) = \int \tilde{f}\, d\mu_2$ for all $f \in M$. Thus,

$$\Gamma(f) = \Gamma_1(f) - \Gamma_2(f) = \int \tilde{f}\, d\mu_1 - \int \tilde{f}\, d\mu_2 = \int \tilde{f}\, d(\mu_1 - \mu_2) \tag{2.21}$$

for all $f \in M$, i.e. φ is well defined. Injectivity of φ directly follows from Equation (2.20) since $\Gamma_1 \neq \Gamma_2$, $\Gamma_1, \Gamma_2 \in V_1$, implies $\int \tilde{f}\, d\mu_1 \neq \int \tilde{f}\, d\mu_2$ for all $f \in M$ with $\Gamma_1(f) \neq \Gamma_2(f)$ and μ_1, resp. μ_2, satisfying Equation (2.20) for Γ_1, resp. Γ_1. Since, by the Hahn-Jordan Decomposition Theorem, every measure μ of bounded variation can be decomposed into a difference $\mu = \mu_1 - \mu_2$, μ_i measures, $i \in \{1, 2\}$, we obtain surjectivity of φ simply by reading Equation (2.21) from right to left, again using Proposition 2.4.3. Linearity of φ is rather obvious. So, we have shown that φ is a linear isomorphism between the linear spaces V_1 and V_2.
By setting $X := M$ and $V := V_1$ in the subsequent Proposition 2.4.5, it follows immediately that φ also is a homeomorphism between the topological spaces $(V_1, \mathcal{T}_{V_1})$ and $(V_2, \mathcal{T}_{V_2})$.
Finally, the last assertion directly follows from Lemma and 2.4.1 Proposition 2.3.4. □

We now provide the deferred, fairly general proposition used in Theorem 2.4.4[2].

[2]This proposition can also be used to prove that the isomorphism between the linear spaces respectively spanned by the totally monotone set functions and the signed bounded

Proposition 2.4.5 *Let X be a non-empty set and V a linear space of real-valued functions on X. Define*

$$\tilde{X} := \{\tilde{x} : V \to \mathbb{R} \mid \tilde{x}(v) := v(x), x \in X\}, \tag{2.22}$$

$$\tilde{V} := \{\tilde{v} : \tilde{X} \to \mathbb{R} \mid \tilde{v}(\tilde{x}) := \tilde{x}(v), v \in V\}, \tag{2.23}$$

$$\tilde{\tilde{X}} := \{$$ [redacted]

missing. For exact set functions, resp. exact functionals, such examinations haven't even been commenced yet. Therefore, it remains as an open problem to determine the set of extreme points of $C(\mathcal{S})$ for a given M. Additionally, for possible applications of Theorem 2.4.4, it remains as an open problem what condition M has to meet in order to make $\mathrm{ex}(C(\mathcal{S}))$ finite.

Theorem 2.4.4 can be used to construct linear inequality preserving functionals in the following way. After determining the extreme points of a convex set $C(\mathcal{S})$ of linear inequality preserving functionals any element of $C(\mathcal{S})$ can be obtained by assigning weights to all extreme points. There is an analogous situation in Dempster-Shafer theory of evidence where these weights are called "basic probability assignments".

Finally, there is some hope that concepts known in standard measure and probability theory like products and independence may be introduced in a fairly natural way for non-additive, linear inequality preserving set functions via the Möbius type transform presented in this chapter.

Chapter 3

A Berry–Esséen Type Estimate for Lévy's Metric

3.1 Introduction

All results providing an estimate of the speed of convergence to the normal distribution can be classified into four groups, on the one hand by using Kolmogorov's metric or any other (e.g. Lévy's metric), and on the other hand by using characteristic functions in the proof or working on the original space of distribution functions. Except for the case of calculating the distance w.r.t. Lévy's metric without using characteristic functions, all variants can be found in literature. In this chapter, we therefore present some upper estimates for Lévy's metric and, as an application, a Berry–Esséen type estimate for the Central Limit Theorem in Lyapunov's version in terms of Lévy's metric which improves the original one.

Our results are motivated as follows. In probability theory, characteristic functions are often introduced as a purely technical tool mainly to prove the Central Limit Theorem (especially in lectures). By proving the latter directly on the space of distribution functions, we avoid technical steps which could distract from the underlying mathematical ideas. Furthermore, the use of Lévy's metric instead of Kolmogorov's metric has the advantage that the former always metricizes convergence in distribution, whereas the latter only metricizes convergence in distribution to continuous limit distributions (which admittedly is sufficient for the proof of the Central Limit Theorem). Finally, our estimate of the rate of convergence to the normal distribution w.r.t. Lévy's metric is better than the original estimate from Berry by far and even topical estimates w.r.t. Kolmogorov's metric are not better than ours.

Since the use of Lévy's metric is not very common, we provide some historical notes. In 1925, Lévy introduced a metric on the space of distribution functions in a very informal way (cf. Lévy 1925 [25, p. 194 – 195, 199 – 200]). In fact, he defined the distance between two distribution functions to be the Hausdorff distance (without referring to it) between the graphs of these distribution functions in $\mathbb{R}^2$ after filling the jumps with line segments. In 1937, Lévy put his definition in concrete terms in a footnote (cf. Lévy 1937 [26, p. 241]) and, in two different versions, in a note (cf. Fréchet 1937 [17, p. 333 – 334]). Unfortunately, the most intuitive and least technical parts of Lévy's work regarding his metric (cf. Lévy 1925 [25, p. 194 – 195] and Fréchet 1937 [17, p. 333 – 334]) are almost completely disregarded in literature. Hence it is not surprising that some facts concerning Lévy's metric – like the relation to the Hausdorff distance (cf. Zolotarev 1997 [41, p. 64]) – have been reinvented. By mostly presenting a definition without its simple geometrical interpretation in literature, Lévy's metric mainly got the status of being a curiosity, at best good enough for exercises in probability books.

This section is organized as follows. In Section 2, Lévy's metric is defined and elementary properties are proved. We also state some geometrical interpretations of Lévy's metric. Relations between Lévy's metric, Fan's metric, and Kolmogorov's metric are shown in Section 3. Some of these relations can be used to improve estimates of Lévy's metric and to show easily the well-known fact that stochastic convergence implies convergence in distribution. In Section 4, we provide some new estimates for Lévy's metric only using the absolute moments of the random variables involved. One of these is used to estimate the rate of convergence to the normal distribution w.r.t. Lévy's metric and we compare this result with existing ones.

3.2 Definition and basic properties

Let $(\Omega, \mathcal{A}, P)$ denote a probability space and for any random variable X let F_X denote the distribution function of X, i.e. $F_X(x) = P(X \leq x)$. $E(X)$ will denote the expected value of X, $V(X)$ will denote the variance of X.

Proposition 3.2.1 *For two random variables $X, Y : \Omega \to \mathbb{R}$, let $d_L(X,Y)$ be defined by*

$$d_L(X,Y) := \inf\Big\{h \geq 0 \mid F_X(x) \leq F_Y(x+h) + h, \\ F_Y(x) \leq F_X(x+h) + h \;\forall x \in \mathbb{R}\Big\}. \quad (3.1)$$

Then d_L is a pseudo-metric on the space of random variables.

Definition 3.2.2 *The pseudo-metric d_L is called **Lévy's metric**.*

The pseudo-metric d_L has an intuitive geometrical interpretation. Both conditions in the definition of Lévy's metric can be rewritten as $F_Y(x+h) \geq F_X(x) - h$ and $F_Y(x-h) \leq F_X(x) + h$ for all $x \in \mathbb{R}$. If F_Y is continuous then, by the Mean Value Theorem, these conditions guarantee that $d_L \leq h$ if and only if F_Y meets every square $S_{x,h}$ with corners at $(x-h, F_X(x)+h)$ and $(x+h, F_X(x)-h)$ for all $x \in \mathbb{R}$. Generally, $d_L \leq h$ if and only if the completed graph $\overline{F_Y}$ of the distribution function F_Y, being defined as a subset of $\mathbb{R}^2$ by $\overline{F_Y} := \{(x,y) \in \mathbb{R}^2 \mid y \in [\lim_{x' \nearrow x} F_Y(x'), \lim_{x' \searrow x} F_Y(x')]\}$, meets every square $S_{x,h}$, $x \in \mathbb{R}$. This condition is equivalent to the claim that the Hausdorff distance (w.r.t. the metric d on $\mathbb{R}^2$, defined by $d((x_1,x_2),(y_1,y_2)) := \max\{|x_1 - y_1|, |x_2 - y_2|\}$), between the sets $\overline{F_X}$ and $\overline{F_Y}$,

$$d_{\text{Hausdorff}}(\overline{F_X}, \overline{F_Y}) = \max\left\{\max_{\bar{x}\in\overline{F_X}} \min_{\bar{y}\in\overline{F_Y}} d(\bar{x},\bar{y}), \max_{\bar{y}\in\overline{F_Y}} \min_{\bar{x}\in\overline{F_X}} d(\bar{x},\bar{y})\right\}, \quad (3.2)$$

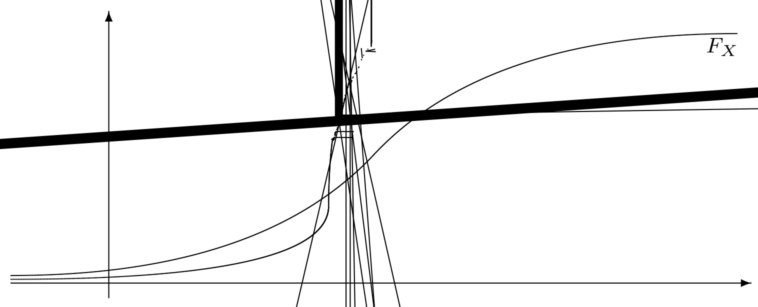
is smaller than or equal to h. Therefore, Lévy's metric can be expressed as the Hausdorff distance on the set of completed graphs of distribution functions, i.e.

$$d_L(X,Y) = d_{\text{Hausdorff}}(\overline{F_X}, \overline{F_Y}). \quad (3.3)$$

Proof of Proposition 3.2.1. By taking a close look at the definition of d_L, we see $d_L \in [0,1]$, especially non-negativity of d_L. Obviously, $d_L(X,X) = 0$. Symmetrie holds by definition. For the proof of the triangular inequality, suppose $h_1 > d_L(X,Y), h_2 > d_L(Y,Z)$. Then for all $x \in \mathbb{R}$

$$F_X(x) \leq F_Y(x+h_1) + h_1 \leq F_Z(x + (h_1+h_2)) + (h_1+h_2)$$

and analogously $F_Z(x) \leq F_X(x + (h_1+h_2)) + (h_1+h_2)$, i.e. $d_L(X,Z) \leq h_1 + h_2$. Hence $d_L(X,Z) \leq d_L(X,Y) + d_L(Y,Z)$. □


F_X

Part (c) is already known (cf. Zolotarev 1967 [40, Lemma 1]) but our proof is shorter and easier. The parts (a) and (b) can intuitively be seen by interpreting Lévy's metric geometrically as the Hausdorff distance between the corresponding completed distribution functions since the geometrical relation of the distribution functions w.r.t. each other does not change by the given operations, translation and reflection w.r.t. the point $(0, \frac{1}{2})$. Nevertheless, a formal proof will be given below.

Proof.

(a) This follows directly from the definition of Lévy's metric and from $F_{x+c}(x) = F_X(x - c)$.

(b) We prove $h > d_L(X,Y)$ implies $h \geq d_L(-X,-Y)$ and $h > d_L(-X,-Y)$ implies $h \geq d_L(X,Y)$. Suppose $h > d_L(X,Y)$. Then there exists an $\varepsilon > 0$ such that $F_X(x) \leq F_Y(x + h - \varepsilon) + h - \varepsilon$ and $F_Y(x) \leq F_X(x + h - \varepsilon) + h - \varepsilon$ for all $x \in \mathbb{R}$. Furthermore,

$$\begin{aligned} & F_X(x) \leq F_Y(x + h - \varepsilon) + h - \varepsilon \\ \Leftrightarrow\ & P(-X \geq -x) \leq P(-Y \geq -x - h + \varepsilon) + h - \varepsilon \\ \Leftrightarrow\ & P(-X < -x) \geq P(-Y < -x - h + \varepsilon) - h + \varepsilon \\ \Leftrightarrow\ & P(-X < -x + h) + h \geq P(-Y < -x + \varepsilon) + \varepsilon \\ \Rightarrow\ & F_{-Y}(-x) \leq F_{-X}(-x + h) + h. \end{aligned}$$

Analogously, we conclude $F_{-X}(-x) \leq F_{-Y}(-x + h) + h$ from $h > d_L(X,Y)$, i.e. $h > d_L(X,Y)$ implies $h \geq d_L(-X,-Y)$. The reverse direction, i.e. $h > d_L(-X,-Y)$ implies $h \geq d_L(X,Y)$, can be obtained in the same way.

(c) Let $h_i := d_L(X_i, Y_i)$, $i = 1, 2$. Then

$$\begin{aligned} F_{X_1+X_2}(x) &= \int F_{X_1}(x - y)\, dF_{X_2}(y) \\ &\leq \int F_{Y_1}(x - y + h_1) + h_1\, dF_{X_2}(y) \\ &= \int F_{X_2}(x - y + h_1)\, dF_{Y_1}(y) + h_1 \\ &\leq \int F_{Y_2}(x - y + h_1 + h_2) + h_2\, dF_{Y_1}(y) + h_1 \\ &= F_{Y_1+Y_2}(x + h_1 + h_2) + h_1 + h_2 \end{aligned}$$

Analogously, we obtain $F_{Y_1+Y_2}(x) \le F_{X_1+X_2}(x + h_1 + h_2) + h_1 + h_2$ and thus the desired inequality.

(d) From the triangular inequality follows

$$d_L(X,Y) \le d_L(X-Y,0) + d_L(Y,Y) = d_L(X-Y,0).$$

Since $F_{|X-Y|} \le F_{X-Y}$ we find for every $h \ge d_L(|X-Y|,0)$

(i) $F_0(x) \le F_{|X-Y|}(x+h) + h \le F_{X-Y}(x+h) + h \quad \text{for all } x \in \mathbb{R}.$

The inequality

(ii) $F_{X-Y}(x) \le F_0(x+h) + h \quad \text{for all } x \ge -h$

is trivially true.
To prove

(iii) $F_{X-Y}(x) \le F_0(x+h) + h \quad \text{for all } x < -h,$

we calculate, using $F_{|X-Y|}(h) \ge F_0(0) - h = 1 - h$,

$$\begin{aligned}
\sup_{x<-h} F_{X-Y}(x) &= \sup_{x>h} P(Y - X \ge x) \\
&\le \sup_{x>h} P(|X-Y| \ge x) \\
&= 1 - \inf_{x>h} P(|X-Y| < x) \\
&= 1 - F_{|X-Y|}(h) \\
&\le h \\
&= F_0(x+h) + h.
\end{aligned}$$

□

3.3 Relations between Lévy's metric and other probability metrics

Probability metrics are commonly introduced to metricize different types of convergence. In this section, we mention Fan's metric and Kolmogorov's metric and enlighten their relations to Lévy's metric.

It is well-known that Lévy's metric metricizes convergence in distribution (cf. e.g. Galambos 1988 [18, Section 4.3]), i.e. for a sequence of random variables X_n holds

$$F_{X_n}(x) \to F_X(x) \; \forall \text{ continuity points } x \text{ of } F_X \iff d_L(X_n, X) \to 0. \quad (3.4)$$

Another probability metric which will turn out to be nicely related to Lévy's metric is *Fan's metric* d_F which is defined on the set of random variables by (cf. Fan 1944 [16] or Dudley 1989 [13, p. 226])

$$d_F(X,Y) := \inf \big\{ h \in \mathbb{R} \mid P(|X-Y| > h) \le h \big\}. \quad (3.5)$$

It is well-known (cf. e.g. Dudley 1989 [13, Theorem 9.2.2]) that this metric metricizes stochastic convergence, i.e.

$$\forall \, \varepsilon > 0 : \; P(|X_n - X| > \varepsilon) \to 0 \iff d_F(X_n, X) \to 0. \quad (3.6)$$

A third metric of interest in our context is

$$d_K(X,Y) := \|F_X - F_Y\|_\infty, \quad (3.7)$$

sometimes referred to as Kolmogorov's metric. As it can easily be shown from the definition, Kolmogorov's metric metricizes convergence in distribution if and only if the limit distribution function is continuous.

In the following proposition, we collect some relations between Lévy's metric, Fan's metric and Kolmogorov's metric. Part (c) is well-known and Part (d) has already been stated (without proof) by Zolotarev (cf. Zolotarev 1997 [41, p. 65]).

Proposition 3.3.1 *Let X, Y be random variables. Then*

(a) $d_F(X,Y) = d_L(|X-Y|, 0)$.

(b) $d_L \le d_F$.

(c) $d_L \le d_K$.

(d) $d_K(X,Y) \le (1 + \|F'_X\|_\infty) \cdot d_L(X,Y)$ *if* F_X *is differentiable.*

Proof.

(a) One easily verifies that the inequalities in the definition of $d_L(|X - Y|, 0)$ hold for all $h > d_F(X,Y)$ and do not hold for all $h < d_F(X,Y)$.

(b) This follows directly from (a) and Proposition 3.2.4 (d).

(c) Elementary calculations show that the inequalities in the definition of d_L hold for all $h > d_K(X,Y)$.

(d) Let $x \in \mathbb{R}$ and suppose $F_Y(x) > F_X(x)$. Then, by differentiability of F_X, $F_X(x+d_L(X,Y)) \leq F_X(x)+d_L(X,Y)\cdot\|F_X'\|_\infty$ and, by definition of Lévy's metric and Lemma 3.2.3, $F_Y(x) \leq F_X(x + d_L(X,Y)) + d_L(X,Y)$. Thus $F_Y(x) \leq F_X(x) + (1 + \|F_X'\|_\infty) \cdot d_L(X,Y)$.
Analogously, we obtain $F_Y(x) \geq F_X(x) - (1 + \|F_X'\|_\infty) \cdot d_L(X,Y)$ in the case $F_Y(x) < F_X(x)$ from $F_X(x - d_L(X,Y)) \geq F_X(x) - d_L(X,Y) \cdot \|F_X'\|_\infty$ and $F_Y(x) \geq F_X(x - d_L(X,Y)) - d_L(X,Y)$. □

These results have important implications. First, from Proposition 3.3.1 (b) directly follows the well-known fact that stochastic convergence implies convergence in distribution. Second, Proposition 3.3.1 (b) can and will in Corollary 3.4.8 be used to adopt upper estimates for Fan's metric between two random variables as some w.r.t. Lévy's metric. Third, Proposition 3.3.1 (c) and (d) will help to compare the rate of convergence of a sequence of random variables to the standard normal distribution when one is given w.r.t. Kolmogorov's metric and the other in terms of Lévy's metric.

We conclude this section with stating an upper estimate for Fan's metric between two random variables which in some cases will improve estimates w.r.t. Lévy's metric in the way remarked in the preceding paragraph. This estimate has the advantage that it only uses the variances of the random variables.

Proposition 3.3.2 *Let X, Y be two random variables with $E(X) = E(Y)$. Then*

$$d_F(X,Y) \leq \sqrt[m+1]{E(|X-Y|^m)} = \left(\|X-Y\|_m\right)^{\frac{m}{m+1}} \tag{3.8}$$

for all $m \in \mathbb{N}$ Additionally, if X, Y are independent then

$$d_F(X,Y) \leq \sqrt[3]{V(X) + V(Y)}. \tag{3.9}$$

Proof. First, suppose $d_F(X,Y) > 0$. From the definition of Fan's metric follows $d_F(X,Y) - \varepsilon < 1 - F_{|X-Y|}(d_F(X,Y) - \varepsilon)$ for every $\varepsilon \in]0, d_F(X,Y)[$. By the Generalized Chebyshev Inequality and by $E(X) = E(Y)$, we get $1 - F_{|X-Y|}(d_F(X,Y) - \varepsilon) \leq E(|X-Y|^m)(d_F(X,Y) - \varepsilon)^{-m}$. This implies $d_F(X,Y) - \varepsilon < \sqrt[m+1]{E(|X-Y|^m)}$ and since $\varepsilon \in]0, d_F(X,Y)[$ was chosen arbitrarily, we obtain $d_F(X,Y) \leq \sqrt[m+1]{E(|X-Y|^m)}$. This inequality obviously also holds for $d_F(X,Y) = 0$. Now suppose X and Y are independent. Then

$$d_F(X,Y) \leq \sqrt[3]{V(X-Y)} = \sqrt[3]{V(X) + V(Y)}.$$

□

3.4 Main Results

The Central Limit Theorem in Lyapunov's version states that whenever

$$\frac{1}{\sigma^3(S_n)} \sum_{i=1}^{n} E(|X_i|^3) \xrightarrow[n\to\infty]{} 0 \tag{3.10}$$

holds for a sequence $(X_n)_{n\in\mathbb{N}}$ of independent identically distributed random variables with $E(X_n) = 0$ for all $n \in \mathbb{N}$ and having finite second and third absolute moments then the normalized partial sum $\frac{S_n}{\sigma(S_n)}$, $S_n := \sum_{i=1}^{n} X_i$, is asymptotically standard normal distributed. The Berry–Esséen Theorem then provides an estimate of the rate of convergence of the distributions of the sequence of partial sums to the standard normal distribution in terms of Kolmogorov's metric depending on the converging term in Formula (3.10) (cf. e.g. Berry 1941 [3, Theorem 5]),

$$d_K\left(\frac{S_n}{\sigma(S_n)}, Y\right) \leq 3.6 \cdot \sqrt[4]{\frac{1}{\sigma^3(S_n)} \sum_{i=1}^{n} E\left(|X_i|^3\right)}, \tag{3.11}$$

with Y being standard normal distributed. Obviously, the Berry–Esséen Theorem implies the Central Limit Theorem.

In this section, we show that there is a Berry–Esséen type theorem for Lévy's metric which, compared to the standard theorem, yields better estimates. To achieve this objective, we provide a class of upper estimates for Lévy's metric all referring to some absolute moments of the random variables.

We adopt a method of proving the Central Limit Theorem directly on the set of distribution functions from Huber (cf. Huber 1975 [22, p. 49 – 53]) which evidently originates from Lindeberg (cf. Lindeberg 1922 [27]). Huber estimated the difference between F_{S_n} and the distribution function of the standard normal distribution without using a probability metric. We incorporate this method in Lemma 3.4.2, Proposition 3.4.3 and Theorem 3.4.9 but by adding the use of Lévy's metric we attach importance to the measurement of the speed of convergence to the normal distribution. Furthermore, the class of estimates for Lévy's metric presented here depends on the use of sufficiently often differentiable approximations of indicator functions of the type $x \mapsto 1_{]-\infty,x_0]}(x)$, $x_0 \in \mathbb{R}$, and we show optimal choices of such functions in Proposition 3.4.4 and Proposition 3.4.5.

As a preliminary result, we start with a special version of Taylor's Theorem and the Fundamental Theorem of Calculus. For a natural number m, denote by $\mathcal{C}_b^m(\mathbb{R})$ the linear space of bounded, real-valued functions on $\mathbb{R}$ having m bounded continuous derivatives. Furthermore, denote by $\mathcal{F}^m(\mathbb{R})$ the linear subspace of $\mathcal{C}_b^{m-1}(\mathbb{R})$ consisting of all functions $f : \mathbb{R} \to \mathbb{R}$ with $f^{(m)}(x)$ existing for all $x \in \mathbb{R}$ except for a finite subset A_f of $\mathbb{R}$ and with $\|f^{(m)}\|_\infty := \|f^{(m)}|_{\mathbb{R}\setminus A_f}\|_\infty < \infty$.

Lemma 3.4.1

(a) Fundamental Theorem of Calculus
Let $f \in \mathcal{F}^1(\mathbb{R})$ and $x_0 \in \mathbb{R}$. Then

$$f(x) = f(x_0) + \int_{x_0}^{x} f'(t)\,dt \tag{3.12}$$

holds for every $x \in \mathbb{R}$.

(b) Taylor's Theorem
Let $f \in \mathcal{F}^m(\mathbb{R})$ and $x_0 \in \mathbb{R}$. Then

$$f(x) = \sum_{k=0}^{m-1} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + R_m(x_0, x) \tag{3.13}$$

holds for all $x \in \mathbb{R}$ with

$$\|R_m(x_0, x)\|_\infty \leq \frac{\|f^{(m)}\|_\infty}{m!}|x - x_0|^m. \tag{3.14}$$

Proof.

(a) If there isn't any discontinuity point of f' between x_0 and x the result holds by the classical Fundamental Theorem of Calculus. Now suppose there is exactly one discontinuity point a of f' between x_0 and x, w.l.o.g. suppose $x_0 < a < x$. Then, applying the classical Fundamental Theorem of Calculus and continuity of f, for sufficiently small $\varepsilon > 0$,

$$\begin{aligned} f(x) &= \big[f(x) - f(a+\varepsilon)\big] + \big[f(a+\varepsilon) - f(a-\varepsilon)\big] + \big[f(a-\varepsilon)f(x_0)\big] \\ &= \int_{a+\varepsilon}^{x} f'(t)\,dt + \big[f(a+\varepsilon) - f(a-\varepsilon)\big] + \int_{x_0}^{a-\varepsilon} f'(t)\,dt \\ &\xrightarrow[\varepsilon\to 0]{} \int_{x_0}^{x} f'(t)\,dt. \end{aligned}$$

If x_0 or x is a discontinuity point of f' itself this result remains true by boundedness of f', resp. continuity of f. By induction, we obtain the general result.

(b) For a given $f \in \mathcal{F}^m(\mathbb{R})$, denote by $A_f \subset \mathbb{R}$ the set of real numbers x for which $f^{(m)}(x)$ does not exists. Furthermore, denote by α the minimal distance between two elements in A_f, $\alpha := \min\{|y-z| \mid y, z \in A_f\}$. Define the sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{C}_b^m(\mathbb{R})$ in the following way. Set

$$f_n^{(m)}(x) := f^{(m)}(x)$$

if $\min_{y\in A_f} |x-y| \geq \frac{\alpha}{2n}$ and

$$f_n^{(m)}(x) := \lambda f^{(m)}(y - \tfrac{\alpha}{2n}) + (1-\lambda) f^{(m)}(y + \tfrac{\alpha}{2n})$$

if $x = \lambda(y - \frac{\alpha}{2n}) + (1-\lambda)(y + \frac{\alpha}{2n})$ for some $y \in A_f$ with $|x - y| < \frac{\alpha}{2n}$ and $\lambda \in [0,1]$. Furthermore, set recursively $f_n^{(k)}(x) := f^{(k)}(0) + \int_0^x f_n^{(k-1)}(t)\,dt$ for each $k = m-1, \dots, 0$. By construction, $\|f_n^{(m)}\|_\infty \leq \|f^{(m)}\|_\infty$. From (a) follows

$$\|f_n^{(m-1)} - f^{(m-1)}\|_\infty \leq |A_f| \cdot \tfrac{1}{2} \cdot \big(2 \cdot \|f^{(m-1)}\|\big) \cdot \big(2\tfrac{\alpha}{2n}\big) = 2 \cdot |A_f| \cdot \|f^{(m-1)}\| \tfrac{\alpha}{2n}.$$

Hence, for every $x \in \mathbb{R}$, the term $|f_n^{(k)}(x) - f^{(k)}(x)|$, $k \in \{0, \dots, m-1\}$, can be estimated only using $2 \cdot |A_f| \cdot \|f^{(m-1)}\| \frac{\alpha}{2n}$, k, and $|x|$. Thus, $f_n^{(k)}$

converges pointwise to $f_n^{(k)}$. Applying Taylor's Theorem for every f_n and using pointwise convergence of $f_n^{(k)}$ to $f^{(k)}$, $k = 0, .., m-1$, yields the desired results. □

Now we come to a fundamental lemma in this section.

Lemma 3.4.2

(a) Let $f \in \mathcal{F}^m(\mathbb{R})$, $m \in \mathbb{N}$ and let $X_1, \dots, X_n$, $Y_1, \dots, Y_n$ be pairwise independent random variables with $E(X_i^k) = E(Y_i^k)$, $i = 1, \dots, n$, $k = 1, \dots, m-1$. Then with $S_n := \sum_{i=1}^n X_i$ and $T_n := \sum_{i=1}^n Y_i$

$$|E(f(S_n)) - E(f(T_n))| \leq \frac{\|f^{(m)}\|_\infty}{m!} \left(\sum_{i=1}^n E(|X_i|^m) + \sum_{i=1}^n E(|Y_i|^m) \right).$$

(b) If especially $n = 1$ then (a) also holds if X_1 and Y_1 are dependent.

Proof. By Lemma 3.4.1,

$$f(X_1 + \cdots + X_n) = \sum_{k=1}^{m-1} \frac{f^{(k)}(S_{n-1})}{k!} X_n^k + R_m(S_{n-1}, S_n) \tag{3.15}$$

with

$$R_m(S_{n-1}, S_n) \leq \frac{\|f^{(m)}\|_\infty}{m!} |X_n|^m.$$

Integrating both sides of Equation (3.15) yields (using independence in the case $n > 1$)

$$E(f(S_n)) = \sum_{k=1}^{m-1} \frac{1}{k!} E(f^{(k)}(S_{n-1})) E(X_n^k) + E(R_m(S_{n-1}, S_n))$$

and with $E(X_i^k) = E(Y_i^k)$ follows

$$\begin{aligned} &\Big| E(f(S_{n-1} + X_n)) - E(f(S_{n-1} + Y_n)) \Big| \\ &\qquad \leq \quad \frac{\|f^{(m)}\|_\infty}{m!} \Big(E(|X_n|^m) + E(|Y_n|^m) \Big). \end{aligned} \tag{3.16}$$

Using

$$\begin{aligned}
&\left|E(f(S_n)) - E(f(T_n))\right| \\
\leq\ & \sum_{i=0}^{n-1} \left| E\left(f\left(\sum_{j=1}^{n-i} X_j + \sum_{j=n-i+1}^{n} Y_j\right)\right) - E\left(f\left(\sum_{j=1}^{n-i-1} X_j + \sum_{j=n-i}^{n} Y_j\right)\right)\right|
\end{aligned}$$

and the corresponding analogous version of (3.16), we get the desired results of (a) and (b). □

The subsequent proposition provides the announced class of upper estimates for Lévy's metric.

Proposition 3.4.3

(a) Let $f \in \mathcal{F}^m(\mathbb{R})$, $m \in \mathbb{N}$, with

$$\begin{array}{ll} f(x) = 1 & \text{if } x \leq 0\,, \\ f(x) \in [0,1] & \text{if } 0 < x < 1\,, \\ f(x) = 0 & \text{if } x \geq 1\,. \end{array} \tag{3.17}$$

Let $X_1, \ldots, X_n$, $Y_1, \ldots, Y_n$ be pairwise independent random variables with $E(X_i^k) = E(Y_i^k)$, $i = 1, \ldots, n$, $k = 1, \ldots, m-1$. Define $S_n := \sum_{i=1}^n X_i$ and $T_n := \sum_{i=1}^n Y_i$. Then

$$d_L(S_n, T_n) \leq \sqrt[m+1]{\frac{\|f^{(m)}\|_\infty}{m!}\left(\sum_{i=1}^n E(|X_i|^m) + \sum_{i=1}^n E(|Y_i|^m)\right)}\,. \tag{3.18}$$

(b) If especially $n = 1$ then (a) also holds if X_1 and Y_1 are dependent.

Proof. Define $f_{x_0,h} : \mathbb{R} \to \mathbb{R}$ by $f_{x_0,h}(x) := f(\frac{x-x_0}{h})$. To prove (a), resp. (b), we use Lemma 3.4.2 (a), resp. (b), and obtain

$$\begin{aligned} F_{S_n}(x_0) &= E(1_{]-\infty,x_0]} \circ S_n) \\ &\leq E(f_{x_0,h} \circ S_n) \\ &\leq E(f_{x_0,h} \circ T_n) + \frac{\|f^{(m)}\|_\infty}{h^m m!} \left(\sum_{i=1}^n E(|X_i|^m) + \sum_{i=1}^n E(|Y_i|^m) \right) \\ &\leq E(1_{]-\infty,x_0+h]} \circ T_n) + \frac{\|f^{(m)}\|_\infty}{h^m m!} \left(\sum_{i=1}^n E(|X_i|^m) + \sum_{i=1}^n E(|Y_i|^m) \right) \\ &= F_{T_n}(x_0 + h) + h^{-m} \cdot \frac{\|f^{(m)}\|_\infty}{m!} \left(\sum_{i=1}^n E(|X_i|^m) + \sum_{i=1}^n E(|Y_i|^m) \right). \end{aligned}$$

With

$$\tilde{h} := \sqrt[m+1]{\frac{\|f^{(m)}\|_\infty}{m!} \left(\sum_{i=1}^n E(|X_i|^m) + \sum_{i=1}^n E(|Y_i|^m) \right)}$$

follows $F_{S_n}(x_0) \leq F_{T_n}(x_0 + \tilde{h}) + \tilde{h}$ and due to symmetry we also obtain $F_{T_n}(x_0) \leq F_{S_n}(x_0 + \tilde{h}) + \tilde{h}$. By the definition of d_L, we get (a), resp. (b). □

The main feature of Proposition 3.4.3 is that Lévy's metric can be estimated only using absolute moments of the random variables involved. By choosing f in this proposition in an optimal way, i.e. minimizing $\|f^{(m)}\|_\infty$ for a given m, we will be able to provide a rate of convergence to the normal distribution, i.e. providing the Berry–Esséen type theorem for Lévy's metric.

We now put Proposition 3.4.3 (a) in concrete terms for $m \in \{1, 2, 3\}$. The occurring regularities of the form of the functions used in the proof gives rise for conjecturing that this result also holds for every natural number m (cf. Conjecture 3.4.6).

Proposition 3.4.4 *Let $X_1, \dots, X_n$, $Y_1, \dots, Y_n$ be pairwise independent random variables satisfying $E(X_i^k) = E(Y_i^k)$ with $i = 1, \dots, n$, $k = 1, \dots, m-1$ and $m \in \{1, 2, 3\}$. Then*

$$d_L(S_n, T_n) \leq \sqrt[m+1]{\frac{4^{m-1}}{m} \left(\sum_{i=1}^n E(|X_i|^m) + \sum_{i=1}^n E(|Y_i|^m) \right)}. \tag{3.19}$$

Proof. Define $f_1, f_2, f_3 : \mathbb{R} \to [0,1]$ by

$$\begin{aligned}
f_1(x) &:= 1 - \int_0^x 4^0 \cdot 0! \cdot 1_{[0,1[}(t_1)\, dt_1\,, \\
f_2(x) &:= 1 - \int_0^x \int_0^{t_1} 4^1 \cdot 1! \cdot \left(1_{[0,\frac{1}{2}[} - 1_{[\frac{1}{2},1[}\right)(t_2)\, dt_2\, dt_1\,, \qquad (3.20)\\
f_3(x) &:= 1 - \int_0^x \int_0^{t_1} \int_0^{t_2} 4^2 \cdot 2! \cdot \left(1_{[0,\frac{1}{4}[\cup[\frac{3}{4},1[} - 1_{[\frac{1}{4},\frac{3}{4}[}\right)(t_3)\, dt_3\, dt_2\, dt_1\,.
\end{aligned}$$

By Lemma 3.4.1 (b), $f_i \in \mathcal{F}^i(\mathbb{R})$ and $\|f_i^{(i)}\|_\infty = 4^{i-1} \cdot (i-1)!$. Applying Proposition 3.4.3 (a) finishes the proof. □

The subsequent proposition answers the question of optimality of the result given in the preceding proposition.

Proposition 3.4.5 *The functions f_i, $i = 1,2,3$, defined in (3.20) satisfy $\|f_i^{(i)}\|_\infty = \inf\{\|f^{(i)}\|_\infty \mid f \in \mathcal{F}^i(\mathbb{R})$ with (3.17)$\}$. Therefore, Estimate (3.19) is the best possible concretion of Estimate (3.18).*

Proof. Suppose $g_1 \in \mathcal{F}^1(\mathbb{R})$ satisfies the Condition (3.17). Then

$$0 = g_1(1) = g_1(0) + \int_0^1 g_1'(t_1)\, dt_1 \geq g_1(0) + \int_0^1 -\|g_1'\|_\infty\, dt_1 = 1 - \|g_1'\|_\infty,$$

i.e. $\|g_1'\|_\infty \geq 1$, hence f_1 is an optimal function.
Now suppose $g_2 \in \mathcal{F}^2(\mathbb{R})$ satisfies the Condition (3.17). W.l.o.g., we can assume that g_2 satisfies the symmetric property $g_2(t) = 1 - g_2(1-t)$ since otherwise we use the function $\overline{g_2} \in \mathcal{F}^2(\mathbb{R})$, defined by

$$\overline{g_2}(t) := \tfrac{1}{2} \cdot g_2(t) + \tfrac{1}{2} \cdot (1 - g_2(1-t)).$$

It satisfies (3.17), $\overline{g_2}(t) = 1 - \overline{g_2}(1-t)$ and $\|\overline{g_2}''\|_\infty \leq \|g_2''\|_\infty$. Then, using symmetry of g_2, $g_2(0) = 1$ and $g_2'(0) = 0$,

$$\begin{aligned}
\tfrac{1}{2} = g_2(\tfrac{1}{2}) &= g_2(0) + \int_0^{\frac{1}{2}} \left(g_2'(0) + \int_0^{t_1} g_2''(t_2)\, dt_2\right) dt_1 \\
&\geq 1 + \int_0^{\frac{1}{2}} \int_0^{t_1} -\|g_2''\|_\infty\, dt_2\, dt_1 \\
&= 1 + \int_0^{\frac{1}{2}} -\|g_2''\|_\infty \cdot t_1\, dt_1 \\
&= 1 - \tfrac{1}{8} \cdot \|g_2''\|_\infty,
\end{aligned}$$

i.e. $\|g_2''\|_\infty \geq 4$, hence f_2 is an optimal function.

Finally, suppose $g_3 \in \mathcal{F}^3(\mathbb{R})$ satisfies the Condition (3.17). From the symmetric property of g_3, $g_3(t) = 1 - g_3(1-t)$ follows $g_3'(t) = g_3'(1-t)$ and $g_3''(t) = -g_3''(1-t)$, hence $g_3''(\frac{1}{2}) = 0$. Furthermore, $g_3''(t) \geq -\|g_3'''\|_\infty \cdot t$, $t \in [0, \frac{1}{4}]$, and $g_3''(t) \geq \|g_3'''\|_\infty \cdot (t - \frac{1}{2})$, $t \in [\frac{1}{4}, \frac{1}{2}]$. Therefore,

$$\begin{aligned}
\tfrac{1}{2} = g_3(\tfrac{1}{2}) &= g_3(0) + \int_0^{\frac{1}{2}} \Big(g_3'(0) + \int_0^{t_1} g_3''(t_2)\, dt_2 \Big)\, dt_1 \\
&= 1 + \int_0^{\frac{1}{4}} \int_0^{t_1} g_3''(t_2)\, dt_2\, dt_1 + \int_{\frac{1}{4}}^{\frac{1}{2}} \int_0^{t_1} g_3''(t_2)\, dt_2\, dt_1 \\
&\geq 1 + \int_0^{\frac{1}{4}} \int_0^{t_1} -\|g_3'''\|_\infty \cdot t_2\, dt_2\, dt_1 \\
&\quad + \int_{\frac{1}{4}}^{\frac{1}{2}} \Big(\int_0^{\frac{1}{4}} -\|g_3'''\|_\infty \cdot t_2\, dt_2 + \int_{\frac{1}{4}}^{t_1} -\|g_3'''\|_\infty \cdot (t_2 - \tfrac{1}{2})\, dt_2 \Big)\, dt_1 \\
&= 1 - \tfrac{1}{64} \cdot \|g_3'''\|_\infty,
\end{aligned}$$

i.e. $\|g_3'''\|_\infty \geq 32$, hence f_3 is an optimal function. □

The Estimate (3.19) has also been proved by the author to be valid for some more natural numbers m. Optimality of Estimate (3.19) has also been proved for $m = 4$. This gives rise to formulate the subsequent conjecture.

Conjecture 3.4.6 *Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ be pairwise independent random variables satisfying $E(X_i^k) = E(Y_i^k)$, $i = 1, \ldots, n$, $k = 1, \ldots, m-1$ and $m \in \mathbb{N}$. Then*

$$d_L(S_n, T_n) \leq \sqrt[m+1]{\frac{4^{m-1}}{m} \left(\sum_{i=1}^n E(|X_i|^m) + \sum_{i=1}^n E(|Y_i|^m) \right)}. \tag{3.21}$$

Before turning to our main theorem, we give two rather simple applications of Proposition 3.4.4.

For a random variable X, denote by $M(X)$ a median of X, i.e.

$$M(X) \in \big[\sup\{x \in \mathbb{R} \mid F_X(x) \leq \tfrac{1}{2}\}, \inf\{x \in \mathbb{R} \mid F_X(x) \geq \tfrac{1}{2}\} \big], \tag{3.22}$$

and by $\tau(X) := E(|X - M(X)|)$ the average absolute deviation from the median. In insurance mathematics, a multiple $\alpha\tau(X)$ of $\tau(X)$, $\alpha > 0$, has been suggested as a risk loading since the premium principle $E(X)+\alpha\tau(X)$ can be represented as a (non-additive) Choquet integral (cf. Denneberg 1994 [11, Exercise 5.4]) having favorable properties for applications. In situations like this one, where the volatility parameter $\tau(X)$ is used, the following corollary of Proposition 3.4.4 may be of interest.

Corollary 3.4.7 *Let X, Y be two random variables with $M(X) = M(Y)$. Then*

$$d_L(X,Y) \leq \sqrt{\tau(X)+\tau(Y)}. \tag{3.23}$$

Proof. Since, by Proposition 3.2.4 (a), $d_L(X,Y) = d_L(X-MX, Y-MX) = d_L(X-MX, Y-MY)$, the statement directly follows from Proposition 3.4.4 for $m = 1$. □

For $m = 2$, Proposition 3.4.4 gets the subsequent form.

Corollary 3.4.8 *Let X, Y be two random variables with $E(X) = E(Y)$. Then*

$$d_L(X,Y) \leq \sqrt[3]{2\Big(V(X)+V(Y)\Big)}. \tag{3.24}$$

Additionally, if X, Y are independent then

$$d_L(X,Y) \leq \sqrt[3]{V(X)+V(Y)}. \tag{3.25}$$

The last assertion directly follows from Proposition 3.3.2 using Proposition 3.3.1 (b).

In his 1967 paper, Zolotarev proved[1] a weaker upper estimate of Lévy's metric than given in (3.25) (cf. Zolotarev 1967 [40, Lemma 2]),

$$d_L(X,Y) \leq \sqrt[3]{4\max\Big\{V(X), V(Y)\Big\}}. \tag{3.26}$$

As the main application of Proposition 3.4.4 we now state a Berry–Esséen type estimate of the rate of convergence to the normal distribution in terms of Lévy's metric.

[1] This proof contains significant gaps or some errors since the use of Chebychev's Inequality seems to be applied by mistake for negative values.

Theorem 3.4.9 *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of independent random variables with $E(X_n) = 0$, $V(X_n) = \sigma_n^2 > 0$ and $E(|X_n|^3)$ finite. Furthermore, let Y be a standard normal distributed random variable which is independent of X_n for all $n \in \mathbb{N}$. Then*

$$d_L\left(\frac{S_n}{\sigma(S_n)}, Y\right) \le 1.93 \cdot \sqrt[4]{\frac{1}{\sigma^3(S_n)} \sum_{i=1}^{n} E\left(|X_i|^3\right)}. \tag{3.27}$$

Proof. Let (Y_n) be a sequence of independent standard normal distributed random variables. Successively using that $\sum_{i=1}^{n} \frac{\sigma(X_i)}{\sigma(S_n)} Y_i$ is standard normal distributed, Proposition 3.4.4 with $m = 3$, $\sigma^3(X_i) \le E(|X_i|^3)$ (Jensen's Inequality) and $E(|Y_i|^3) = \sqrt{\frac{8}{\pi}}$ for all $i \le n$, we obtain

$$\begin{aligned} d_L\left(\frac{S_n}{\sigma(S_n)}, Y\right) &= d_L\left(\frac{S_n}{\sigma(S_n)}, \sum_{i=1}^{n} \frac{\sigma(X_i)}{\sigma(S_n)} Y_i\right) \\ &\le \sqrt[4]{\frac{16}{3\sigma^3(S_n)} \left(\sum_{i=1}^{n} E\left(|X_i|^3\right) + \sum_{i=1}^{n} \sigma^3(X_i) E\left(|Y_i|^3\right)\right)} \\ &\le \sqrt[4]{\frac{16}{3\sigma^3(S_n)} \left(\sum_{i=1}^{n} E\left(|X_i|^3\right)\left(1 + E\left(|Y_i|^3\right)\right)\right)} \\ &\le \sqrt[4]{\frac{16}{3\sigma^3(S_n)} \left(1 + \sqrt{\frac{8}{\pi}}\right)\left(\sum_{i=1}^{n} E\left(|X_i|^3\right)\right)} \\ &\le 1.93 \cdot \sqrt[4]{\frac{1}{\sigma^3(S_n)} \sum_{i=1}^{n} E\left(|X_i|^3\right)}. \end{aligned}$$ □

A natural question arising now is how to compare the standard Berry–Esséen estimate of the rate of convergence w.r.t. Kolmogorov's metric to the one obtained above. Using $\|F_Y'\|_\infty = (\sqrt{2\pi})^{-1}$ for a standard normal distributed random variable Y, Proposition 3.3.1 (c), (d) and Theorem 3.4.9 together yield that an estimate in terms of Kolmogorov's metric,

$$d_K\left(\frac{S_n}{\sigma(S_n)}, Y\right) \le C \cdot \sqrt[4]{\frac{1}{\sigma^3(S_n)} \sum_{i=1}^{n} E\left(|X_i|^3\right)}, \tag{3.28}$$

is better than the one in Theorem 3.4.9 if $C < 1.93$, incomparable if $C \in [1.93, 2.70]$, and worse if $C > 2.70$. Since $C = 3.6$ in Berry's original estimate (cf. Inequality (3.11)), our estimate is an improvement. Breiman has mentioned, that there exist unpublished calculations giving bounds as low as $C = 2.05$ (cf. Breiman 1992 [6, p. 184]). This bound is incomparable to our result, but this also means that it is not better than ours.

3.5 Conclusions

It remains as an open problem to prove Conjecture 3.4.6. Although all relevant cases of this conjecture, i.e. those cases actually used in this chapter, have been proved in Proposition 3.4.4, it would be a nice result. Another task remaining to do is to provide a rate of convergence for sequences of independent distributed random variables (having certain additional properties) converging in distribution to a Poisson distributed random variable. Such a result cannot be expressed in terms of Kolmogorov's metric since the limit distribution is not continuous and would therefore expose the advantages of Lévy's metric.

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List of Symbols

d_K	Kolmogorov's metric, page 47
d_L	Lévy's metric, page 43
$\mathrm{ex}(X)$	set of extreme points of a convex set X, page 32
$E(X)$	expected value of the random variable X, page 42
$\mathcal{F}^m(\mathbb{R})$	linear subspace of $\mathcal{C}_b^{m-1}(\mathbb{R})$ consisting of all functions $f : \mathbb{R} \to \mathbb{R}$ with $f^{(m)}$ exists except for finitely many $x \in \mathbb{R}$, page 50
F_X	distribution function of the random variable X, page 42
Γ_*	natural extension of Γ, page 7, 8, 10, 25
$\Gamma_\bullet$	natural exactification of Γ, page 8, 10, 25
M	non-empty subset of $B(2^\Omega)$, page 2, 30
$M(X)$	median of the random variable X, page 56
$\vert \cdot \vert$	Minkowski-functional characterizing exactifiable functionals, page 4
$\Vert \cdot \Vert$	Minkowski-functional characterizing exact functionals, page 4
$\Vert \cdot \Vert_{\mathrm{op}}$	operator norm, page 30
Ω	non-empty set, page 2, 30
$\mathcal{S}$	non-empty set of finite sets in $\mathbb{R} \times M$, page 32
$\mathcal{T}$	topology of pointwise convergence on $B(M)$, page 30
$\tau(X)$	average absolute deviation of the random variable X from $M(X)$, page 56
$V(X)$	variance of the random variable X, page 42

Index